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A Closed-Form Formula for Calculating The Fourier Transform of Time-Limited Signals

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Abstract

A new method for calculating the Fourier transform of a function is developed. This method is based on the generalization of the jump formula which is used in calculating the Fourier series of periodic functions. The developed formula is a very user-friend and handy equation. Besides the application of developed routine for calculating the Fourier transform, it is very useful in evaluating the convergence rate of the Fourier transform of the functions. It is believed that the proposed relation will very soon be appeared in Engineering Mathematics text books and communities.

Key words: Fourier Transform; jump algorithm; convergence rate; Fourier analysis; Fourier formulation

1. Introduction

Fourier series is one of the best methods for analysis of the spectral contents of signals. Given a time-domain periodic signal [1-3], Fourier series gives the amplitude and phase of the different harmonics. These Fourier coefficients are usually calculated by integration or by inner product relations. However, there is a summation formula which gives these coefficients without integration [4]. This relation is based on the jumps of the signal or jumps of its derivatives. In other words, the signal can be regenerated by knowing its jumps. This formula relates the convergence rate of Fourier series of a periodic signal to the smoothness degree of the signal, or in other words, to its jump values. The importance of this method urged the author to seek for a similar routine for calculating Fourier transform of a signal.

Here, at first, the author presents the exponential version of the jump formula for Fourier series, and then, he derives a new formula for calculating the Fourier transform of a function versus its jumps. Then he argues the application of the proposed formula in analysis of the convergence rate of the Fourier transform. The author predicts that this formulation leads to derive more useful similar relations. Up to the latest information of the author, there is no similar argument in the literatures.

The paper is arranged as follows. In the next section, a review of Fourier series and Fourier transform is presented. The jump formula in the exponential format is derived in section 3. In section 4, the derivation of the desired formula together with some supporting examples is presented, and in section 5, application of the developed formula in analysis of convergence is argued. Section 6 summarizes the advantages of the proposed formula. The paper finishes with a short conclusion.

2. Fourier Analysis

2.1. Fourier series

A periodic signal with period $T = 2l$ can be expressed by an infinite series as [1-4]:

$$g(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} \left(a_n \cos\frac{n\pi x}{l} + b_n \sin\frac{n\pi x}{l} \right) \quad (1)$$

in which the a_n's andb_n's are known as the Fourier or Euler coefficients of function $g(x)$, and are given by:

$$a_n = \frac{1}{l}\int_{-l}^{l} \cos\left(\frac{n\pi x}{l}\right) g(x)dx \quad (2)$$

$$b_n = \frac{1}{l}\int_{-l}^{l} \sin\left(\frac{n\pi x}{l}\right) g(x)dx \quad (3)$$

The exponential or complex versions of the (1) to (3) are:

$$g(x) = \sum_{n=-\infty}^{\infty} c_n \exp\left(\frac{in\pi x}{l}\right) \qquad (4)$$

$$c_n = \frac{1}{2l}\int_{-l}^{l} e^{\frac{-in\pi x}{l}} g(x)dx \quad (-\infty < n < \infty) \qquad (5)$$

which are simply obtained by applying Euler formals, i.e. $\cos(x) \pm i\sin(x) = e^{\pm ix}$, to relations (1) to (3).

2.2. Fourier transform

To study aperiodic signals in frequency domain, the Fourier series is converted to Fourier transform. In other words, the Fourier transform is assumed as the limit of Fourier series when $l \to \infty$. In this case, the relation between a signal $(g(x))$ and its transform $(G(f))$ is given by the following pair:

$$G(f) = \int_{-\infty}^{\infty} g(x) e^{-i2\pi fx} dx \qquad (6)$$

$$g(x) = \int_{-\infty}^{\infty} G(f) e^{i2\pi fx} df \qquad (7)$$

It must be noted that for both Fourier series and Fourier transform, the signals must satisfy some convergence theorems such as Dirichlet's condition [4], and we assume that this is the case. Meanwhile, it is noticeable that there are other definitions for Fourier transform e.g. the kernel $e^{-i2\pi fx}$ may be replaced by either of $e^{i2\pi fx}$, $e^{-i\omega x}$, or $\frac{1}{\sqrt{2\pi}}e^{-i\omega x}$. However, all of these notations lead to the same result.

2.3. The relation between Fourier transform and Fourier series

If a signal $g(x)$ is zero beyond $[-l, l]$ as shown in Figure 1.a., its Fourier transform i.e. the equation (6) can be restated as:

$$G(f) = \int_{-l}^{l} g(x) e^{-i2\pi fx} dx \qquad (8)$$

or in a closed form:

Comparing (5) and (8) leads to:

$$c_n = \frac{1}{2l} G\left(\frac{n}{2l}\right) = f_0 G(nf_0) \qquad (9)$$

in which $f_0 = 1/2l$ is the fundamental frequency of the periodic version of $g(x)$ shown in Figure 1.b. Equation (9) shows the relation between Fourier series coefficients and the Fourier transform.

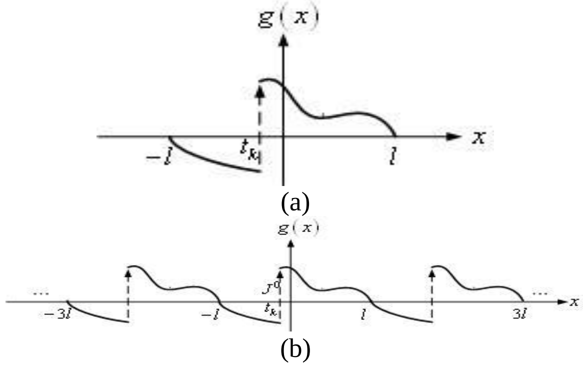

Figure 1. An aperiodic signal (a) and its periodic version (b).

3. Derivation of jump formula for Fourier series

Consider the signal and its derivatives are discontinuous at some point t_k. Let's assume that the value of the jump for the m^{th} order derivative i.e. $g^{(m)}(x)$ is denoted by $J^{(m)}$. Now separation of (5) into two integrals gives:

$$c_n = \frac{1}{2l}\left\{\int_{-l}^{t_k^-} e^{\frac{-in\pi x}{l}} g(x)dx + \int_{t_k^+}^{l} e^{\frac{-in\pi x}{l}} g(x)dx\right\} \qquad (10)$$

Replacing $\omega = n\pi / l$ for the sake of simplicity and using Kronecker algorithm leads to:

$$c_n = \frac{1}{2l}\left\{\frac{e^{-in\omega t_k}}{(in\omega)}\left\{g(t_k^-) - g(t_k^+)\right\} + \frac{e^{-in\omega t_k}}{(in\omega)^2}\left\{g'(t_k^-) - g'(t_k^+)\right\} + \ldots\right\} \qquad (11)$$

$$c_n = \frac{1}{T} e^{-in\omega t_k} \sum_{m=0}^{M} \frac{J_k^{(m)}}{(in\omega)^{m+1}} \quad (12)$$

In (12), we have used the notation of the m^{th} order jump of $g(x)$ as $J^{(m)} = g^{(m)}(t_k^+) - g^{(m)}(t_k^-)$.

If there are J discontinuous points for each period of the signal and its derivatives, (12) can be generalized as:

$$c_n = \frac{1}{T} \sum_{k=1}^{J} e^{-jn\omega t_k} \sum_{m=0}^{M} \frac{J_k^{(m)}}{(jn\omega)^{m+1}} \quad (13)$$

It is noticeable that the relation (13) is developed by the author. In other words, although the trigonometric venison of (13) is found in the literatures, this more useful relation is not found in the mathematics books, journals and in the internet.

Example: *Calculating the Fourier series of periodic pulse.* Table 1 shows the jump values of the function $g(t) = rep_T\left(\Pi(\frac{2t}{T})\right)$ and its derivatives. Using this table, it is seen that the Fourier series coefficient of the pulse function, can be calculated by:

$$c_n = \frac{1}{T}\left\{\frac{1e^{jn\pi/2}}{(jn2\pi/T)^1} + \frac{-1e^{-jn\pi/2}}{(jn2\pi/T)^1}\right\} = \frac{1}{2}\mathrm{sinc}\left(\frac{n}{2}\right)$$

which is equivalent to $a_n = \mathrm{sinc}\left(\frac{n}{2}\right)$ and $a_0 = 1$. The function together with its Fourier series sum has been shown in Figure 2.

Table 1. The jump values of periodic pulse in one period.

	$t_1 = -\frac{T}{4}$		$t_2 = +\frac{T}{4}$	
$g(t)$	$J_1 = 1$	$\omega t_1 = -\frac{\pi}{2}$	$J_2 = -1$	$\omega t_2 = \frac{\pi}{2}$
$g'(t)$	0		0	

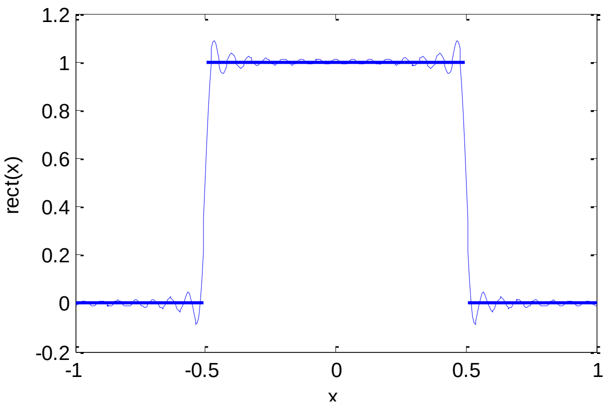

Figure 2. A symmetric pulse and its Fourier sum.

4. Derivation of the proposed jump formula

The key point which led the author to derive a formula which expresses the Fourier transform versus the jumps of the signal is merging equations (9) and (13). Comparing these two equations we have:

$$f_0 G(nf_0) = \frac{1}{T} \sum_{k=1}^{J} e^{-jn\omega t_k} \sum_{m=0}^{M} \frac{J_k^{(m)}}{(jn\omega)^{m+1}} \quad (14)$$

which gives:

$$G(f) = \sum_{k=1}^{J} e^{-j\omega t_k} \sum_{m=0}^{M} \frac{J_k^{(m)}}{(j\omega)^{m+1}} \quad (15)$$

Of course, if one prefers to use the notation $G(\omega)$ instead of $G(f)$, he or she can use $G(\omega)$ in the RHS of the above formula. Therefore, we have proposed the following theorem:

Theorem: Assume a function $g(x)$ satisfies the convergence conditions. If there is an l so that beyond $[-l, l]$, $g(x)$ is zero, then the Fourier transform of $g(x)$ defined by $G(\omega) = \int_{-\infty}^{\infty} g(x) e^{-i\omega x} dx$ can be expressed by the following series:

$$G(\omega) = \sum_{k=1}^{J} e^{-j\omega t_k} \sum_{m=0}^{M} \frac{J_k^{(m)}}{(j\omega)^{m+1}} \quad (16)$$

The meanings of the parameters of this formula are:

t_k: The k^{th} discontinuous point of the function or its derivatives.

J: Overall number of discontinuous points of the function and its derivatives.

M: A number so that $g^{(m>M)}(x) = 0$. This value may be infinity.

$J_k^{(m)}$: The value of jump of $g^{(m)}(x)$ at point t_k.

Example one: *Calculating the Fourier transform of a pulse.* Consider the symmetric pulse or rectangular function $\Pi(t)$. As shown in Figure 2, it has two jumps at $t_1 = \frac{-1}{2}$ and $t_2 = \frac{1}{2}$ with values of $J_1 = 1$ and $J_2 = -1$. In this case, $M = 0$ and $J = 2$. Using (15), we have:

$$G(\omega) = \frac{e^{-j\omega\frac{-1}{2}} - e^{-j\omega\frac{1}{2}}}{j\omega} = \mathrm{sinc}\left(\frac{\omega}{2\pi}\right) = \mathrm{sinc}(f)$$

which is the same value found in the most Engineering mathematics text books.

Example two: *Calculating the Fourier transform of a finite cosine function.* Consider the function shown in Figure 3. This function is smooth in all points but two jump points at $t_1 = \frac{-1}{2}$ and $t_2 = \frac{1}{2}$. The values of the jumps of the function and the jumps of its derivatives at these two points are given in Table 2.

Table 2. The jump values of a windowed cosine function

	$J_k^{(0)}$	$J_k^{(1)}$	$J_k^{(2)}$	$J_k^{(3)}$	$J_k^{(4)}$	J_k^{5}	$J_k^{(6)}$
$t_1 = \frac{-1}{2}$	0	$+\pi$	0	$-\pi^3$	0	$+\pi^5$	0
$t_2 = \frac{1}{2}$	0	$+\pi$	0	$-\pi^3$	0	$+\pi^5$	0

For this example, $J = 2$ and M approaches to infinity. Using (15), we have:

$$G(f) = e^{+j\pi f}\left\{\frac{+\pi}{(j2\pi f)^2} + \frac{-\pi^3}{(j2\pi f)^4} + \frac{+\pi^5}{(j2\pi f)^6} + \cdots\right\} + e^{-j\pi f}\left\{\frac{+\pi}{(j2\pi f)^2} + \frac{-\pi^3}{(j2\pi f)^4} + \frac{+\pi^5}{(j2\pi f)^6} + \cdots\right\}$$

The functions in each of the brackets show a geometric progression, therefore we get:

$$G(f) = \frac{-e^{+j\pi f}\pi}{(2\pi f)^2}\left\{\frac{1}{1-(2f)^{-2}}\right\} - \frac{e^{-j\pi f}\pi}{(2\pi f)^2}\left\{\frac{1}{1-(2f)^{-2}}\right\} = \left\{\frac{2\pi\cos(\pi f)}{\pi^2 - (2\pi f)^2}\right\}$$

This result is easily validated by direct calculation or by using modulation theorem i.e. $\cos(\pi t)\Pi(t)$. Using a simple Matlab code we have calculated the Fourier transform of $\cos(\pi t)\Pi(t)$. The results together with the function are shown in Figure 3.

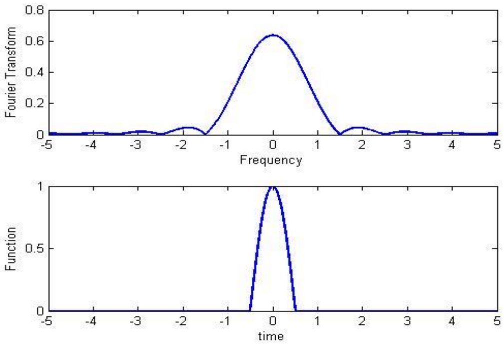

Figure 3. A finite cosine function and its Fourier sum.

5. Convergence analysis of Fourier Transform using the proposed formula

Equation (15) as its predecessor in Fourier series can be used to analyze the rate of convergence of the function with frequency. This formula expresses that if the function is discontinuous, then the dominant mode in the expansion of $G(\omega) = \sum_{k=1}^{\infty} a_k \omega^{-k}$ is ω^{-1}. In other words, for a discontinuous function, we get $a_1 \neq 0$. Similarly, we get the following results:

a. If $g(t)$ and its successive derivatives up to the m^{th} order (i.e. $g^{(m)}(t)$) are continuous then the dominant mode in the expansion of $\mathrm{G}(\omega) = \sum_{k=1}^{\infty} a_k \omega^{-k}$ is $\omega^{-(m+1)}$.
b. If a function such as $g(t)$ is smoother than another function, let's say $h(t)$, then the dominant mode in the expansion of $\mathrm{G}(\omega)$ has higher power than that of $\mathrm{H}(\omega)$.

Example one: *Analyzing the convergence rate of the Fourier transform of a pulse.* Here we want to analyze the convergence rate of the function $\Pi(\mathrm{t})$, as shown in Figure 4. This function has two discontinuities in $t = \frac{\pm 1}{2}$. The proposed formula and its results show that this function has a ω^{-1} term. This is compatible with the behavior of the sinc function, i.e. $\mathrm{G}(\omega) = \mathrm{sinc}\frac{\omega}{2\pi}$.

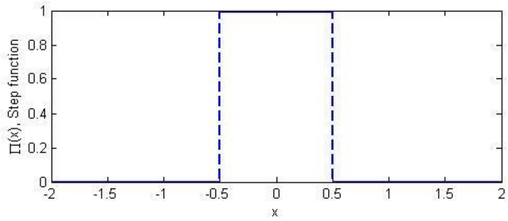

Figure 4. A rectangular pulse function.

Example two: *Analyzing the convergence rate of the Fourier transform of finite cosine function.* Here we want to analyze the convergence rate of the function $\cos(\pi t)\Pi(t)$. As shown in Figure 3, this function is continuous everywhere in $-\infty < t < \infty$. However, its first derivative is discontinuous, therefore the dominant mode in the expansion of its Fourier transform is ω^{-2}. This is the result we gained in the previous section.

6. Advantages of the proposed routine

Although the proposed formula is restricted to time-limited signals, it has a few advantages which makes it very useful and applicable to many functions. First, the proposed formula is very simple and easy to be used for calculating the Fourier transform of functions. Second, the proposed formula is a closed form one. Third, due to analytic nature of the proposed relation, it is much faster with respect to the discrete algorithms e.g. DFT or FFT [5]. Fourth, the obtained formula is an exact relation for the Fourier transform; therefore it is more accurate compared to the discrete Fourier schemes. Fifth, it can be used to search for many properties of the transform as well as proving some new theorems. This is what we used for proving convergence theorem. Sixth, it is a new bridge and connection between Fourier series and Fourier transform. These and some other benefits which let us to predict that the proposed relation is going to be appeared in Mathematics communities in near future.

7. Conclusion

A new method for calculating the Fourier transform of a signal has been developed. The proposed method is based on the generalization of the jump formula. The developed formula can be easily used to calculate the Fourier transform of time-limited signals, without any need to integration. Besides the application of developed routine for calculating the Fourier transform, it can be employed in evaluating the convergence rate of the Fourier transform of the functions. It should be noted that the proposed routine presents an exact formula for the Fourier transform. Therefore, it cannot be compared to FFT and similar algorithms in which an approximate function is seeked. The author believes that the proposed relation will very soon be appeared in text books and Engineering Mathematics communities.

8. References

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