REPUBLIC OF TURKEY
FIRAT UNIVERSITY
THE GRADUATE SCHOOL OF NATURAL AND APPLIED SCIENCE

SOLUTION OF INVERSE PROBLEMS FOR STURM-LIOUVILLE PROBLEM ENERGY-DEPENDENT POTENTIAL

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Master Thesis

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Program: Analysis and Functions Theory
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DECEMBER-2016

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SUMMARY

This thesis consists of six sections incliuding the references.

In first and second sections introduction and some definitions and theorems used in the main sections are given, respectively.

In third section, Ambarzumyan's theorem for Sturm-Liouville equation with spectral parameter in boundary condition is expressed and proved.

In fourth section, we consider Sturm-Liouville problem energy dependent potential with spectral parameter in boundary condition. We give asymptotic formulas for eigenvalues and also, we proved that if the spectrums are the same for two problem, then $\int_0^\pi (q - \widetilde{q})dx = 0$. Finally, we show that if the spectrum is $\sigma(0)$, then $q = 0$ almost everywhere on $[0,\pi]$.

Finally, in the fifth section, we give some conclutions.

Keywords : Ambarzumyan's theorem, Spectrum, Sturm-Liouville problem, Potential function

LIST OF SYMBOLS

$\mathbb{N}$	:	Set of the netural numbers
$\mathbb{R}$	:	Set of the real numbers
$\mathbb{C}$	:	Set of the complex numbers
$L_2[a, b]$	:	the set of all squared integrable real-valued functions on $[a, b]$.
λ	:	Eigenvalue

1.INTRODUCTION

Generally, inverse problem for Sturm-Liouville problem was introduced by the Greeks in the 19th century. However, the beginning of the inverse problem for the Sturm-Liouville operator has been demonstrated in studies conducted by Russian astrophysicist Ambartsumyan the early 20th century. In this study Ambartsumyan showed that [1] just one spectrum is enough to find the potential as zero. In fact, this idea was not true in general. Because just a single spectrum is not enough to determine the potential function as uniquely. In this study we gave the idea that the spectrum can be obtained by using knowledge of the eigenvalues of the potential function.

In spectral theory some important results have been obtained for [1-5].

$$-y''(x) + q(x)\,y(x) = \lambda y(x) \quad , \quad x \in [0,\pi]$$

which is known as non-dimensional time-dependent Schrödinger equation, where $q(x)$ is potential function and λ is an eigenvalue of the problem. The first study on spectral theory for such operators is in [1]. In later years, many mathematician have studied such these problems [6,7].

Essentially, there are two ways to solve the Sturm-Liouville problem first, the potential function is given in direct problems and then the spectral datas are asked to find us. In the second, given all spectral parameters like eigenvalues, eigenfunctions, etc. and then one asks to find the potential function. When we compare these problems, second is more popular. By the way, after 1980, authors showed that it is possible to find the potential function by using the zeros of eigenfunctions. This is called inverse nodal problem in the literature [7,8].

2. BASIC DEFINITION AND THEOREMS

Definition 2.1 : (Metric Space) Let X be a non-empty set. A metric (or distance)

$$d : X \times X \longrightarrow \mathbb{R}$$

such that , for $\forall x, y, z \in X$ satisfies

(M1) $d(x, y) \geq 0$

(M2) $d(x, y) = 0 \Leftrightarrow x = y$

(M3) $d(x, y) = d(y, x)$

(M4) $d(x, z) \leq d(x, y) + d(y, z)$

Then (X, d) is called metric spaces [9].

Example 2.1 : The set of all functions satisfying the condition $\int (f(x))^2 dx < \infty$ with distance

$$d(f(x), g(x)) = \sqrt{\int (f(x) - g(x))^2 dx}$$

is a metric space $L_2(\mathbb{R})$.

Definition 2.2 : **(Normed space)** Let X be a linear space . A norm on X is real-valued function $\|.\|$ on X defined by

$$\begin{aligned} \|.\| &: X \to \mathbb{R} \\ x &\to \|x\| \end{aligned}$$

satisfying

(N1) $\|x\| \geq 0,\ x \in X.$

(N2) $\|x\| = 0, \Leftrightarrow x = 0$.

(N3) $\forall x \in X$ and for all scalars α, $\|\alpha x\| = |\alpha| \, \|x\|$

(N4) $\forall x, y \in X$, $\|x + y\| \leq \|x\| + \|y\|$

Then $(X , \|.\|)$ defines a normed space [9].

Example 2.2 : L_2 space is a normed space for $X = (x_1, x_2, x_{3,} ..., x_n, ...) \in L_2$ with the norm,

$$\|x\| = \left(\sum_{n=1}^{\infty} |x_n|^2 \right)^{\frac{1}{2}}$$

Definition 2.3: (Operator) For vector spaces X and Y, an operator is defined by $T : X \to Y$.

Definition 2.4: **(Linear operator)** Let X and Y be two normed linear space on the same field of scalars. A transformation defined by

$$T : X \to Y$$

is said to be linear if

i) $T(x + y) = T(x) + T(y)$

ii) $T(\alpha x) = \alpha T(x)$ for all $x, y \in X$ and for all scalars α. A linear transformation

$$T : X \to Y$$

is said to be continuous at $x_0 \in X$ if

$$x_n \to x_0 \Rightarrow T(x_n) \to T(x_0).$$

T is said to be continuous if T is continuous at each point of X. Note that T is continuous if and only if

$$x_n \to x \Longrightarrow T(x_n) \to T(x).$$

Let consider a linear transformation as $T : X \to Y,$ where X and Y are normed spaces. If there exists a constant $c > 0$, so that

$$\|T(x)\| \leq c\,\|x\|\,,\ \forall x \in X.$$

we say that T is a bounded [10].

Definition 2.5: (Hilbert space) Let X be a complex linear space. An inner-product on X defined as

$$\langle\ ,\ \rangle :\ X \times X \to \mathbb{C}$$

which satisfies the following conditions:

1) $\langle x, x\rangle \geq 0$, $\langle x, x\rangle = 0 \Longleftrightarrow x = 0.$

2) $\langle x, y + z\rangle = \langle x, y\rangle + \langle x, z\rangle$.

3) $\langle x, y\rangle = \overline{\langle y, x\rangle},$

4) $\langle \alpha x, y\rangle = \alpha \langle x, y\rangle$ where $x, y, z \in X$ and α is a complex number [9]. We have the following relation between the norm and inner product

$$\|x\| = \sqrt{\langle x, x\rangle}.$$

Example 2.3: Let $L_{2,\rho}\left[a,b\right]$ be a Hilbert space. For $x(t)\in L_{2,\rho}\left[a,b\right]$, if $\rho(t)>0$, we can write that [9]

$$\int_a^b \rho(t)\left|x(t)\right|^2 dt<\infty$$

For $\forall x(t),y(t)\in L_{2,\rho}[a,b]$, the inner product is defined by

$$\langle x,y\rangle=\int_a^b \rho(t)x(t)\overline{y(t)}dt.$$

Example 2.4 : Let as consider the Hilbert space $L_2[a,b]$. In this space, the inner product defines for $\forall f(x),g(x)\in L_2[a,b]$

$$\langle f,g\rangle=\int_a^b f(x)\overline{g(x)}dx.$$

Definition 2.6: **(Eigenvalues, Eigenvectors, Spectrum, Resolvent set).** An eigenvalue of an operator A is a λ such that

$$Ax=\lambda x$$

has a solution $x\neq 0$. Also x is defined as an eigenvector corresponding to λ. The set $\sigma\left(A\right)$ of all eigenvalues of A is called the spectrum of A. Also, we define resolvent as the complement of the spectrum. For an exmple, by direct calculation we can verify that $x_1=\begin{bmatrix}4\\1\end{bmatrix}$ and $x_2=\begin{bmatrix}1\\-1\end{bmatrix}$ are eigenvectors of $A=\begin{bmatrix}5&4\\1&2\end{bmatrix}$ corresponding to the eigenvalues $\lambda_1=6$ and $\lambda_2=1$ of A, respectively. How did we obtain this result, and what can we say about the existence of eigenvalues of a matrix in general? To answer this question, let us first note that

$$Ax=\lambda x \tag{2.1}$$

can be written

$$(A-\lambda I)=0 \tag{2.2}$$

where I is the $n-$rowed unit matrix. This is a homogeneous system of n linear equations in n unknowns $\xi_1,\xi_2,...,\xi_n$, which are the components of x.

The determinant of the coefficients is $\det(A - \lambda I)$ and must be zero in order that (2.2) has a solution $x \neq 0$. This gives the **characteristic equation** of A :

$$\det(A - \lambda I) = \begin{bmatrix} \alpha_{11} - \lambda & \alpha_{12} & \cdots & \alpha_{1n} \\ \alpha_{21} & \alpha_{22} - \lambda & \cdots & \alpha_{2n} \\ \cdot & \cdot & \cdots & \cdot \\ \cdot & \cdot & \cdots & \cdot \\ \cdot & \cdot & \cdots & \cdot \\ \alpha_{n1} & \alpha_{n2} & \cdots & \alpha_{nn} - \lambda \end{bmatrix} = 0. \tag{2.3}$$

We can say that $det(A - \lambda I)$ is **characteristic determinant** of A. By developing, we obtain a polynomial in λ of degree n, the **characteristic polynomial** of A. Equation (2.3) is called the characteristic equation of A.

Let us consider a normed space X and Y is a linear operator from $D(T)$ to X , where $D(T)$ is domain of T. On the other hand, let us considers an operator as

$$T_\lambda = T - \lambda I, \tag{2.4}$$

where λ is a complex number and I is the identity operator on $D(T)$. If T_λ has an inverse, we denote it by $R_\lambda(T)$,

$$R_\lambda(T) = T_\lambda^{-1} = (T - \lambda I)^{-1} \tag{2.5}$$

and call it the resolvent operator of T or simply the **resolvent** of T [9].

Definition 2.7 : (Regular value, Resolvent set, Spectrum). Let $X \neq \{0\}$ be a complex normed space and $T : D(T) \to X$ a linear operator with domain $D(T) \subset X$. A regular value λ of T is a complex number satisfiying

(R1) $R_\lambda(T)$ exists.

(R2) $R_\lambda(T)$ is bounded.

(R3) $R_\lambda(T)$ is defined on a set which is dense.

The resolvent set $\rho(T)$ of T is the set of all regular value λ of T. Its complement $\sigma(T) = \mathbb{C} - \rho(T)$ in the complex plane $\mathbb{C}$ is called the spectrum of T, and $\lambda \in \sigma(T)$ is called a spectral value of T.

The **point spectrum** or **discrete spectrum** $\sigma_p(T)$ is the set so that $R_\lambda(T)$ does not exist. $\lambda \in \sigma_p(T)$ is called an eigenvalue of T.

The **continuous spectrum** $\sigma_c(T)$ is the set so that $R_\lambda(T)$ exists and satisfies R3 not R2.

The **residual spectrum** $\sigma_r(T)$ is the set such that $R_\lambda(T)$ exists (and may be bounded or not) but does not satisfy R3 [9].

Defintion 2.8:(Adjoint Operator) Assume that H be a Hilbert space and $T : H \rightarrow H$ be a bounded linear operator on H. Then the unique operator T^* defined for $\forall x, y \in H$

$$\langle Tx, y \rangle = \langle x, T^* y \rangle ,$$

is called the adjoint of T. The mapping $T \rightarrow T^*$ is called the adjoint operation on H [9].

Definition 2.9:(Self-Adjoint Operator) Considering a linear operator T on H. If $T^* = T$ we say that T is self-adjoint. And this operator satisfies

$$\langle Tx, y \rangle = \langle x, Ty \rangle , \ \forall x, y \in H.$$

The concept of self-adjoint operator is fundamental in mathematical physics [9,11].

Definition 2.10: (Sturm-Liouville Equation) Let consider the following equation

$$c_1(x) \frac{d^2 u}{dx^2} + c_2(x) \frac{du}{dx} + [c_3(x) + \lambda] u = 0 \tag{2.6}$$

where, c_1, c_2, c_3 are coefficients functions depend on x. If we introduce

$$p(x) = e^{\int \frac{c_2(x)}{c_1(x)} dx}, \quad q(x) = \frac{c_3(x)}{c_1(x)} p, \quad s(x) = \frac{1}{c_1(x)} p \tag{2.7}$$

in (2.6) we get

$$\frac{d}{dx} \left(p \frac{du}{dx} \right) + [q + \lambda s] u = 0 \tag{2.8}$$

which is known as the Sturm-Liouville equation in literature. In terms of the self-adjoint operator

$$L = \frac{d}{dx} \left(p \frac{d}{dx} \right) + q \tag{2.9}$$

it can be written briefly

$$L[u] + \lambda s(x) u = 0. \tag{2.10}$$

In the Sturm-Liouville equation (2.8) λ is a parameter independent of x and $p(x), q(x)$ and $s(x)$ are real-valued functions. To insure the existence of solutions, we let $q(x)$

and $s(x)$ to be continuous and $p(x)$ to be continuously differentiable in a closed finite interval $[a, b]$. Generally, the Sturm-Liouville equation is considered the following conditions

$$a_1 u(a) + a_2 u'(a) = 0 \tag{2.11}$$

$$b_1 u(b) + b_2 u'(b) = 0 \tag{2.12}$$

where $a_1, a_2, b_1, b_2 \in \mathbb{R}$. [11]

Defintion 2.11 (O and o) Let consider two functions f, g. If there is a constant C such that $|f(x)| \leq C|g(x)|$, we say that $f = O(g)$ and similarly if $\frac{f}{g} \to 0$, we can say that $f = o(g)$ for $x \to \infty$ [10].

3. AMBARZUMYAN'S THEOREM FOR STURM-LIOUVILLE PROBLEM

In this part, we will consider classical Sturm-Lioville equation with boundary condition depending polynomial spectral parameter [4,5,6]. Firstly, we will give asymptotic formula for eigenvalues in general case. Then, by using these formulas we prove the Classical Ambarzumyan's theorem [6,12]. All of these results were given in [3,4].

We consider the eigenvalue problem

$$-y''(x) + q(x)y(x) = \lambda y(x) \quad \text{on } [0,\pi]\,, \tag{3.1}$$

subject to

$$y'(0) = 0,\ y'(\pi) + f(\lambda)y(\pi) = 0. \tag{3.2}$$

that are different from the classical Sturm-Liouville eigenvalue problem. Because there is a general function depend on eigenvalue in the boundary condition at π [3,4].

In all stages we will assume that $q \in C[0,\pi]$ and

$$f(\lambda) = a_1\sqrt{\lambda} + a_2\sqrt{\lambda}^2 + a_3\sqrt{\lambda}^3 + ... + a_m\sqrt{\lambda}^m \quad , a_m \in \mathbb{R}, a_m \neq 0, m \in Z^+. \tag{3.3}$$

Let's sign the problem $(3.1)\,,(3.2)$ with $B = B(q, f(\lambda))$.

Generally, it is not possible to find the potential q by just one spectrum. But if the potential is symmetric on the interval, one spectrum is enough to find the potential. It is also interesting to expand this problem to the problem which is given with boundary condition depending polynomial spectral parametar [13,14].

Let $\phi_1(x,\lambda)$ be the solution of differential equation

$$-y''(x) + q(x)\,y(x) = \lambda y(x) \tag{3.4}$$

with initial condition

$$y(0) = 1$$

It is well known that [6,7]

$$\phi_1(x,\lambda) = \cos\sqrt{\lambda}x + \int_0^x K(x,t)\cos\sqrt{\lambda}t dt, \tag{3.5}$$

where $K(x,t)$ has derivatives with respect to the variables. Similarly, let $\phi_2(x,\lambda)$ be the solution of the differential equation

$$-y''(x)+q(x)y(x)=\lambda y(x), \tag{3.6}$$

$$y(0)=0,$$

and $\phi_2(x,\lambda)$ can be written [6-7]

$$\phi_2(x,\lambda)=\frac{\sin\sqrt{\lambda}x}{\sqrt{\lambda}}+\frac{1}{\sqrt{\lambda}}\int_0^x K(x,t)\sin\sqrt{\lambda}tdt. \tag{3.7}$$

Furthermore, the kernel of transformation $K(x,t)$ satisfies

$$K(x,x)=\frac{1}{2}\int_0^x q(x)dx. \tag{3.8}$$

Integration by parts yields in (3.5)

$$\phi_1(\pi,\lambda)=\cos\sqrt{\lambda}\pi+K(\pi,\pi)\frac{\sin\sqrt{\lambda}\pi}{\sqrt{\lambda}}-\frac{1}{\sqrt{\lambda}}\int_0^\pi K_t'(\pi,t)\sin\sqrt{\lambda}tdt \tag{3.9}$$

and

$$\phi_1'(\pi,\lambda)=-\sqrt{\lambda}\sin\sqrt{\lambda}\pi+K(\pi,\pi)\cos\sqrt{\lambda}\pi+\int_0^\pi K_x'(\pi,t)\cos\sqrt{\lambda}tdt. \tag{3.10}$$

It is well known that [5,15] λ is an eigenvalues of the problem if and only if

$$\phi_1'(\pi,\lambda)+f(\lambda)\phi_1(\pi,\lambda)=0. \tag{3.11}$$

Then, substituting (3.9) and (3.10) in (3.11), we obtain.

$$\begin{aligned} W(\lambda) &= -\sqrt{\lambda}\sin\sqrt{\lambda}\pi+f(\lambda)\cos\sqrt{\lambda}\pi+K(\pi,\pi)\cos\sqrt{\lambda}\pi \\ &\quad +\int_0^\pi K_x'(\pi,t)\cos\sqrt{\lambda}tdt+\frac{f(\lambda)\sin\sqrt{\lambda}\pi K(\pi,\pi)}{\sqrt{\lambda}} \\ &\quad -\frac{f(\lambda)}{\sqrt{\lambda}}\int_0^\pi K_t'(\pi,t)\sin\sqrt{\lambda}tdt = 0. \end{aligned} \tag{3.12}$$

We will call it as characteristic function for $W(\lambda)$ [6,15].

Now, we are ready to give asymptotics of eigenvalues and uniqueness theorem for $(3.1),(3.2)$. By the way, we will denote the spectrum of the problem (3.1) (3.2) by $\sigma(q)$.

Lemma 3.1 [16,17]

1) If $m=1$ in (3.3)

$$\sqrt{\lambda_n}=n+\frac{\arctan a_1}{\pi}+\frac{\int_0^\pi q(x)\,dx}{2n\pi}+O\left(\frac{1}{n^2}\right). \tag{3.13}$$

2) If $m=2$ in (3.3)

$$\sqrt{\lambda_n}=n+\frac{1}{2}+\frac{\int_0^\pi q(x)\,dx-\frac{2}{a_2}}{(2n+1)\pi}+O\left(\frac{1}{n^2}\right). \tag{3.14}$$

3) If $m\geq 3$ in (3.3)

$$\sqrt{\lambda_n}=n+\frac{1}{2}+\frac{\int_0^\pi q(x)\,dx}{(2n+1)\pi}+O\left(\frac{1}{n^2}\right). \tag{3.15}$$

In particular, for the three subcases there has $0\in\sigma_p(0)$.

Proof : By (3.12), we see that λ is an eigenvalue of the problem (3.1),(3.2) if and only if

$$\begin{aligned}&-\sqrt{\lambda}\sin\sqrt{\lambda}\pi+f(\lambda)\cos\sqrt{\lambda}\pi+K(\pi,\pi)\cos\sqrt{\lambda}\pi\\&+\int_0^\pi K'_x(\pi,t)\cos\sqrt{\lambda}tdt+\frac{f(\lambda)\sin\sqrt{\lambda}\pi K(\pi,\pi)}{\sqrt{\lambda}}\\&-\frac{f(\lambda)}{\sqrt{\lambda}}\int_0^\pi K'_t(\pi,t)\sin\sqrt{\lambda}tdt\ =\ 0.\end{aligned} \tag{3.16}$$

1) For $m=1$, then $f(\lambda)=a_1\sqrt{\lambda}$ by inserting in (3.16) we get

$$\begin{aligned}&-\sqrt{\lambda}\sin\sqrt{\lambda}\pi+a_1\sqrt{\lambda}\cos\sqrt{\lambda}\pi+K(\pi,\pi)\cos\sqrt{\lambda}\pi\\&+\int_0^\pi K'_x(\pi,t)\cos\sqrt{\lambda}tdt+\frac{a_1\sqrt{\lambda}\sin\sqrt{\lambda}\pi K(\pi,\pi)}{\sqrt{\lambda}}\\&-\frac{a_1\sqrt{\lambda}}{\sqrt{\lambda}}\int_0^\pi K'_t(\pi,t)\sin\sqrt{\lambda}tdt\ =0\end{aligned}$$

or

$$\sqrt{\lambda}\sin\sqrt{\lambda}\pi=a_1\sqrt{\lambda}\cos\sqrt{\lambda}\pi+K(\pi,\pi)\cos\sqrt{\lambda}\pi+a_1\sin\sqrt{\lambda}\pi K(\pi,\pi)+O\left(\frac{e^{\left|\operatorname{Im}\sqrt{\lambda}\right|\pi}}{\sqrt{\lambda}}\right).$$

Then, we can obtain

$$\tan\sqrt{\lambda}\pi=a_1+\frac{K(\pi,\pi)}{\sqrt{\lambda}}+\frac{a_1K(\pi,\pi)\tan\sqrt{\lambda}\pi}{\sqrt{\lambda}}+O\left(\frac{1}{\lambda}\right)$$

which implies that

$$\tan\sqrt{\lambda}\pi - a_1 = \frac{\left(1+a_1^2\right)K\left(\pi,\pi\right)}{\sqrt{\lambda}} + O\left(\frac{1}{\lambda}\right).$$

By applying same trigonometric identity

$$\tan\sqrt{\lambda}\pi - a_1 = \tan\left(\sqrt{\lambda}\pi - \arctan a_1\right)\left(1 + a_1\tan\sqrt{\lambda}\pi\right).$$

and

$$\begin{aligned}
\tan\left(\sqrt{\lambda}\pi - \arctan a_1\right) &= \left[1 + a_1\tan\sqrt{\lambda}\pi\right]^{-1}\left[\frac{\left(1+a_1^2\right)K\left(\pi,\pi\right)}{\sqrt{\lambda}} + O\left(\frac{1}{\lambda}\right)\right] \\
&= \left[\frac{1}{1+a_1^2} + O\left(\frac{1}{\sqrt{\lambda}}\right)\right]\left[\frac{\left(1+a_1^2\right)K\left(\pi,\pi\right)}{\sqrt{\lambda}} + O\left(\frac{1}{\lambda}\right)\right] \\
&= \frac{K\left(\pi,\pi\right)}{\sqrt{\lambda}} + O\left(\frac{1}{\lambda}\right).
\end{aligned}$$

Then

$$\begin{aligned}
\arctan\left[\tan\left(\sqrt{\lambda}\pi - \arctan a_1\right)\right] &= \arctan\left[\frac{K\left(\pi,\pi\right)}{\sqrt{\lambda}} + O\left(\frac{1}{\lambda}\right)\right] \\
\sqrt{\lambda}\pi - \arctan a_1 &= n\pi + \frac{K\left(\pi,\pi\right)}{\sqrt{\lambda}} + O\left(\frac{1}{\lambda}\right) \\
\sqrt{\lambda_n} &= n + \frac{\arctan a_1}{\pi} + \frac{K\left(\pi,\pi\right)}{\sqrt{\lambda_n}\pi} + O\left(\frac{1}{\lambda_n}\right),
\end{aligned}$$

$$\begin{aligned}
\sqrt{\lambda_n} &= n\left[1 + \frac{\arctan a_1}{n\pi} + \frac{K\left(\pi,\pi\right)}{\sqrt{\lambda_n}n\pi} + O\left(\frac{1}{n\lambda_n}\right)\right] \\
\frac{\sqrt{\lambda_n}}{n} &= 1 + \frac{\arctan a_1}{n\pi} + \frac{K\left(\pi,\pi\right)}{\sqrt{\lambda_n}n\pi} + O\left(\frac{1}{n\lambda_n}\right). \qquad (*)
\end{aligned}$$

By above we obtain

$$\frac{1}{\sqrt{\lambda_n}} = \frac{1}{n}\left[1 + O\left(\frac{1}{n}\right)\right].$$

Then, inserting this value in $(*)$, we obtain

$$\begin{aligned}
\frac{1}{\sqrt{\lambda_n}} &= \frac{1}{n}\left[1 + \frac{\arctan a_1}{n\pi} + \frac{K\left(\pi,\pi\right)}{\sqrt{\lambda_n}n\pi} + O\left(\frac{1}{n\lambda_n}\right)\right] \\
&= \frac{1}{n} + \frac{\arctan a_1}{n^2\pi} + \frac{K\left(\pi,\pi\right)}{\sqrt{\lambda_n}n^2\pi} + O\left(\frac{1}{n^2\lambda_n}\right) \\
&= \frac{1}{n} + O\left(\frac{1}{n^2}\right).
\end{aligned}$$

Then by using

$$\frac{1}{\sqrt{\lambda_n}} = \frac{1}{n} + O\left(\frac{1}{n^2}\right),$$

we can get

$$\begin{aligned}
\sqrt{\lambda_n} &= n + \frac{\arctan a_1}{\pi} + \frac{K(\pi,\pi)}{n\pi} + O\left(\frac{1}{n^2}\right) \\
&= n + \frac{\arctan a_1}{\pi} + \frac{\int_0^\pi q(x)\,dx}{2n\pi} + O\left(\frac{1}{n^2}\right).
\end{aligned}$$

2).For $m = 2$, By (3.16), we have

$$\begin{aligned}
\cot\sqrt{\lambda}\pi &= -\frac{K(\pi,\pi)}{\sqrt{\lambda}} + \frac{\sqrt{\lambda}}{f(\lambda)} + O\left(\frac{1}{\lambda}\right) \\
&= -\frac{K(\pi,\pi)}{\sqrt{\lambda}} + \frac{\sqrt{\lambda}}{a_m\sqrt{\lambda}^m} + O\left(\frac{1}{\lambda}\right) \\
&= -\frac{K(\pi,\pi)}{\sqrt{\lambda}} + \frac{1}{a_m\sqrt{\lambda}^{m-1}} + O\left(\frac{1}{\lambda}\right)
\end{aligned}$$

and

$$\begin{aligned}
\cot\sqrt{\lambda}\pi &= -\frac{K(\pi,\pi)}{\sqrt{\lambda}} + \frac{\sqrt{\lambda}}{a_2\sqrt{\lambda}^2} + O\left(\frac{1}{\lambda}\right) \\
&= -\frac{K(\pi,\pi)}{\sqrt{\lambda}} + \frac{1}{a_2\sqrt{\lambda}} + O\left(\frac{1}{\lambda}\right) \\
&= \frac{\frac{1}{a_2} - K(\pi,\pi)}{\sqrt{\lambda}} + O\left(\frac{1}{\lambda}\right).
\end{aligned}$$

By using again trigonometric identity

$$\begin{aligned}
\operatorname{arccot}\left(\cot\sqrt{\lambda}\pi\right) &= \operatorname{arccot}\left[\frac{\frac{1}{a_2} - K(\pi,\pi)}{\sqrt{\lambda}} + O\left(\frac{1}{\lambda}\right)\right] \\
\sqrt{\lambda}\pi &= \left(n + \frac{1}{2}\right)\pi + \frac{K(\pi,\pi) - \frac{1}{a_2}}{\sqrt{\lambda}} + O\left(\frac{1}{\lambda}\right) \\
\sqrt{\lambda_n} &= n + \frac{1}{2} + \frac{K(\pi,\pi) - \frac{1}{a_2}}{\sqrt{\lambda}\pi} + O\left(\frac{1}{\lambda}\right).
\end{aligned}$$

Then,

$$\begin{aligned}
\sqrt{\lambda_n} &= \left(n+\frac{1}{2}\right)\left[1+\frac{K(\pi,\pi)-\frac{1}{a_2}}{\sqrt{\lambda_n}\pi\left(n+\frac{1}{2}\right)}+O\left(\frac{1}{n^2}\right)\right] \\
\sqrt{\lambda_n} &= \left(n+\frac{1}{2}\right)\left[1+O\left(\frac{1}{n}\right)\right] \qquad (**)
\end{aligned}$$

$$\frac{\sqrt{\lambda_n}}{n+\frac{1}{2}}=1+O\left(\frac{1}{n}\right)$$

$$\frac{\sqrt{\lambda_n}}{n+\frac{1}{2}}=1+\frac{K(\pi,\pi)-\frac{1}{a_2}}{\sqrt{\lambda_n}\pi\left(n+\frac{1}{2}\right)}+O\left(\frac{1}{n^2}\right).$$

On the other hand, by using $(**)$

$$\frac{1}{\sqrt{\lambda_n}}=\frac{1}{\left(n+\frac{1}{2}\right)\left[1+O\left(\frac{1}{n}\right)\right]},\ \text{for}\ \ n\to\infty,$$

or

$$\frac{1}{1+O\left(\frac{1}{n}\right)}=1+O\left(\frac{1}{n}\right).$$

Then, we can easily write that

$$\begin{aligned}
\frac{1}{\sqrt{\lambda_n}} &= \frac{1}{\left(n+\frac{1}{2}\right)}\left[1+\frac{K(\pi,\pi)-\frac{1}{a_2}}{n\pi\left(n+\frac{1}{2}\right)}+O\left(\frac{1}{n^2}\right)\right] \\
&= \frac{1}{n+\frac{1}{2}}+O\left(\frac{1}{n^2}\right)
\end{aligned}$$

Eventually, we can see that

$$\begin{aligned}
\sqrt{\lambda_n} &= \left(n+\frac{1}{2}\right)+\frac{K(\pi,\pi)-\frac{1}{a_2}}{\left(n+\frac{1}{2}\right)\pi}+O\left(\frac{1}{n^2}\right) \\
&= \left(n+\frac{1}{2}\right)+\frac{\int_0^\pi q(x)\,dx-\frac{2}{a_2}}{(2n+1)\pi}+O\left(\frac{1}{n^2}\right).
\end{aligned}$$

3) For $m=3$, we obtain by (3.16)

$$\begin{aligned}
\cot\sqrt{\lambda}\pi &= -\frac{K(\pi,\pi)}{\sqrt{\lambda}}+\frac{\sqrt{\lambda}}{f(\lambda)}+O\left(\frac{1}{\lambda}\right) \\
&= -\frac{K(\pi,\pi)}{\sqrt{\lambda}}+\frac{\sqrt{\lambda}}{a_m\sqrt{\lambda}^m}+O\left(\frac{1}{\lambda}\right) \\
&= -\frac{K(\pi,\pi)}{\sqrt{\lambda}}+\frac{1}{a_3\lambda}+O\left(\frac{1}{\lambda}\right) \\
&= -\frac{K(\pi,\pi)}{\sqrt{\lambda}}+O\left(\frac{1}{\lambda}\right)
\end{aligned}$$

and

$$\begin{aligned}\sqrt{\lambda_n} &= n+\frac{1}{2}+\frac{K(\pi,\pi)}{\left(n+\frac{1}{2}\right)\pi}+O\left(\frac{1}{\lambda}\right)\\ &= n+\frac{1}{2}+\frac{\int_0^\pi q(x)\,dx}{(2n+1)\pi}+O\left(\frac{1}{n^2}\right).\end{aligned}$$

Finally, if $m>3$ it is easy to see that

$$\sqrt{\lambda_n}=n+\frac{1}{2}+\frac{\int_0^\pi q(x)\,dx}{(2n+1)\pi}+O\left(\frac{1}{n^2}\right).$$

This completes the proof.

Lemma.3.2: [16,17] Let f be Riemann integrable on $[a,b]$. Then

$$\begin{aligned}\lim_{\lambda\to\pm\infty}\int_a^b f(x)\cos(\lambda x)\,dx &= 0\\ \lim_{\lambda\to\pm\infty}\int_a^b f(x)\sin(\lambda x)\,dx &= 0\\ \lim_{\lambda\to\pm\infty}\int_a^b f(x)\,e^{i\lambda x}dx &= 0.\end{aligned}$$

Consider a second eigenvalue problem by changing q with $\widetilde{q}$

$$-y''(x)+\widetilde{q}(x)\,y(x)=\lambda y(x) \tag{3.17}$$

$$y'(0)=0,\ \ y'(\pi)+f(\lambda)\,y(\pi)=0, \tag{3.18}$$

where $\widetilde{q}(x)$ has the same properties of $q(x)$. In here, we want to prove if the spectrums are the coincide, then the potentials are the same.

Theorem 3.1: [16,17] Assume $\sigma_p(q)=\sigma_p(\widetilde{q})$, then $\int_0^\pi[q(x)-\widetilde{q}(x)]\,dx=0$.

Proof: Let's consider λ_n large eigenvalue in $\sigma_p(q)$. Since $\sigma_p(q)=\sigma_p(\widetilde{q})$, it follows $\lambda_n\in\sigma_p(\widetilde{q})$. Thus, we obtain by (3.12)

$$\begin{aligned}&-\sqrt{\lambda_n}\sin\sqrt{\lambda_n}\pi+f(\lambda_n)\cos\sqrt{\lambda_n}\pi+K(\pi,\pi)\cos\sqrt{\lambda_n}\pi\\ &+\int_0^\pi K'_x(\pi,t)\cos\sqrt{\lambda_n}t\,dt+\frac{f(\lambda)\sin\sqrt{\lambda_n}\pi K(\pi,\pi)}{\sqrt{\lambda_n}}\\ &-\frac{f(\lambda)}{\sqrt{\lambda_n}}\int_0^\pi K'_t(\pi,t)\sin\sqrt{\lambda_n}t\,dt \;=\; 0\end{aligned} \tag{3.19}$$

and similarly for the problem $(3.17), (3.18)$

$$\begin{aligned} & -\sqrt{\lambda_n}\sin\sqrt{\lambda_n}\pi + f(\lambda_n)\cos\sqrt{\lambda_n}\pi + \widetilde{K}(\pi,\pi)\cos\sqrt{\lambda_n}\pi \\ & + \int_0^{\pi} \widetilde{K}'_x(\pi,t)\cos\sqrt{\lambda_n}tdt + \frac{f(\lambda)\sin\sqrt{\lambda_n}\pi\widetilde{K}(\pi,\pi)}{\sqrt{\lambda_n}} \\ & -\frac{f(\lambda)}{\sqrt{\lambda_n}}\int_0^{\pi} \widetilde{K}'_t(\pi,t)\sin\sqrt{\lambda_n}tdt \; = \; 0. \end{aligned} \tag{3.20}$$

By subtracting the last equalities we obtain

$$\begin{aligned} & \left[K(\pi,\pi) - \widetilde{K}(\pi,\pi)\right]\cos\sqrt{\lambda_n}\pi + \int_0^{\pi}\left[K_x(\pi,t) - \widetilde{K}_x(\pi,t)\right]\cos\sqrt{\lambda_n}tdt \\ & +\frac{f(\lambda_n)\sin\sqrt{\lambda_n}\pi}{\sqrt{\lambda_n}}\left[K(\pi,\pi) - \widetilde{K}(\pi,\pi)\right] \\ & -\frac{f(\lambda_n)}{\sqrt{\lambda_n}}\int_0^{\pi}\left[K_t(\pi,t) - \widetilde{K}_t(\pi,t)\right]\sin\sqrt{\lambda_n}tdt \; = \; 0 \end{aligned} \tag{3.21}$$

and by lemma (3.2), second and fourth terms goes to zero as $\lambda_n \to \infty$. Then,

$$\left[K(\pi,\pi) - \widetilde{K}(\pi,\pi)\right]\left(\cos\sqrt{\lambda_n}\pi + \frac{f(\lambda_n)}{\sqrt{\lambda_n}}\sin\sqrt{\lambda_n}\pi\right) = 0.$$

1) If $m = 1$ and $n \to \infty$,

$$\begin{aligned} & \left[K(\pi,\pi) - \widetilde{K}(\pi,\pi)\right]\left(\cos\sqrt{\lambda_n}\pi + \frac{a_1\sqrt{\lambda}}{\sqrt{\lambda}}\sin\sqrt{\lambda_n}\pi\right) \; = \; 0 \\ & \left[K(\pi,\pi) - \widetilde{K}(\pi,\pi)\right](\cos(n\pi + \arctan a_1) + a_1\sin(n\pi + \arctan a_1)) \; = \; 0 \\ & \left[K(\pi,\pi) - \widetilde{K}(\pi,\pi)\right][(\cos n\pi\cos(\arctan a_1) - \sin n\pi\sin(\arctan a_1)) \\ & +a_1\sin n\pi\cos(arac\tan a_1) + a_1\cos n\pi\sin(\arctan a_1)] \; = \; 0 \\ & \left[K(\pi,\pi) - \widetilde{K}(\pi,\pi)\right](\cos n\pi\cos(\arctan a_1) + a_1\cos n\pi\sin(\arctan a_1)) \; = \; 0 \\ & (-1)^n\left[K(\pi,\pi) - \widetilde{K}(\pi,\pi)\right][\cos(\arctan a_1) + a_1\sin(\arctan a_1)] \; = \; 0. \end{aligned}$$

We know that

$$\begin{aligned} \cos(\arctan a_1) &= \frac{1}{\sqrt{1+a_1^2}} \\ \sin(\arctan a_1) &= \frac{a_1}{\sqrt{1+a_1^2}}, \end{aligned}$$

then we can write

$$(-1)^n \left[K(\pi,\pi) - \widetilde{K}(\pi,\pi)\right]\left(\frac{1}{\sqrt{1+a_1^2}} + a_1 \frac{a_1}{\sqrt{1+a_1^2}}\right) = 0$$
$$(-1)^n \left[K(\pi,\pi) - \widetilde{K}(\pi,\pi)\right]\left(\sqrt{1+a_1^2}\right) = 0.$$

and we conclude that by (3.8) $\int_0^\pi [q(x) - \widetilde{q}(x)]\,dx = 0.$

2) If $m = 2$, in this case by using (3.21) can be divided both side by $\frac{\sqrt{\lambda_n}}{f(\lambda_n)}$ we obtain

$$\begin{aligned}
&\frac{\sqrt{\lambda_n}}{f(\lambda_n)} \cos\sqrt{\lambda_n}\pi \left[K(\pi,\pi) - \widetilde{K}(\pi,\pi)\right] + \\
&\frac{\sqrt{\lambda_n}}{f(\lambda_n)} \int_0^\pi \left[K_x(\pi,t) - \widetilde{K}_x(\pi,t)\right] \cos\sqrt{\lambda_n} t dt + \sin\sqrt{\lambda_n}\pi \left[K(\pi,\pi) - \widetilde{K}(\pi,\pi)\right] \\
&- \int_0^\pi \left[K_t(\pi,t) - \widetilde{K}(\pi,t)\right] \sin\sqrt{\lambda_n} t dt = 0.
\end{aligned} \tag{3.22}$$

Together with (3.14) and (3.15), and if $m = 2$, then

$$\begin{aligned}
&\frac{\sqrt{\lambda_n}}{a_1\sqrt{\lambda_n} + a_2\lambda_n} \cos\sqrt{\lambda_n}\pi \left[K(\pi,\pi) - \widetilde{K}(\pi,\pi)\right] + \sin\sqrt{\lambda_n}\pi \left[K(\pi,\pi) - \widetilde{K}(\pi,\pi)\right] = 0 \\
&\left[K(\pi,\pi) - \widetilde{K}(\pi,\pi)\right]\left[\frac{\sqrt{\lambda_n}}{a_1\sqrt{\lambda_n} + a_2\lambda_n} \cos\sqrt{\lambda_n}\pi + \sin\sqrt{\lambda_n}\pi\right] = 0 \\
&\left[K(\pi,\pi) - \widetilde{K}(\pi,\pi)\right]\left[\frac{\sqrt{\lambda_n}}{a_1\sqrt{\lambda_n} + a_2\lambda_n} \cos\left(n+\frac{1}{2}\right)\pi + \sin\left(n+\frac{1}{2}\right)\pi\right] = 0 \\
&\left[K(\pi,\pi) - \widetilde{K}(\pi,\pi)\right]\Bigg[\frac{\sqrt{\lambda_n}}{a_1\sqrt{\lambda_n} + a_2\lambda}(\cos n\pi \cos\frac{\pi}{2} - \sin n\pi \sin\frac{\pi}{2}) \\
&+ \sin n\pi \cos\frac{\pi}{2} + \cos n\pi \sin\frac{\pi}{2}\Bigg] = 0
\end{aligned}$$

$$\left[K(\pi,\pi) - \widetilde{K}(\pi,\pi)\right](-1)^n = 0, \tag{3.23}$$

which implies that

$$\int_0^\pi [q(x) - \widetilde{q}(x)]\,dx = 0.$$

For $m \geq 3$,we can easily prove that $\int_0^\pi [q(x) - \widetilde{q}(x)]\,dx = 0$.

Theorem 3.2 :[16,17] Suppose $q \in C[0,\pi]$ and $\sigma_p(q) = \sigma_p(0)$ .Then $q(x) = 0$ on $[0,\pi]$.

Proof : By assumption in the Theorem we, get

$$\int_0^\pi q(x)\,dx = 0. \tag{3.24}$$

We know that 0 is also an eigenvalue of the problem. Let u_1 be the eigenfunction corresponding to the eigenvalue 0. Then, by (3.1) and (3.2),

$$-u_1''(x) + q(x) u_1(x) = 0 \tag{3.25}$$
$$u_1'(0) = u_1'(\pi) = 0.$$

We claim that $u_1(0) \neq 0$ and $u_1(\pi) \neq 0$. Otherwise, if $u_1(0) = 0 = u'(0)$ or $u_1(\pi) = 0 = u_1'(\pi)$ we have a contrary to theorem of ordinary differential equation. That is, $u_1(x)$ is the eigenfunction corresponding to the eigenvalue zero.

Now, we want to show that $u_1(x)$ has at most finitely many isolated zeros in the interval $[0,\pi]$. Otherwise, let $\widehat{x}$ be an accumulation point of zeros of $u_1(x)$, then $u_1(\widehat{x}) = u_1'(\widehat{x}) = 0$ holds true which is impossible from the uniqueness theorem of the solution of initial value problems. Let's denote $G = \{x_1, x_2, ..., x_k : x_i \in (0,\pi), u_1(x_i) = 0\}$, then mass $G = 0$. From (3.25) we see that if $u_1(x_i) = 0$, then $u_1''(x_i) = 0$, but $u_1'(x_i) \neq 0$. Thus, the value of the function $\frac{u''(x)}{u'(x)}$ at $x = x_i$ is finite.

Collecting all of these and by (3.24) and (3.25), we obtain

$$\int_0^\pi \frac{u''(x)}{u(x)} dx = \int_0^\pi q(x)\, dx = 0,$$

or

$$\left.\frac{u_1'(x)}{u_1(x)}\right|_{x=0}^{\pi} + \int_0^\pi \left|\frac{u_1'(x)}{u_1(x)}\right|^2 dx = 0$$

which implies that

$$\int_0^\pi \left|\frac{u_1'(x)}{u_1(x)}\right|^2 dx = 0.$$

Thus, we obtain $u_1(x) = c$ that is solution of (3.25). Inserting this solution in the equation, we get

$$-c'' + q(x) c = 0 \text{ on } [0,\pi],$$

thus $q(x) = 0$.

4. MBARZUMYAN THEOREM FOR STURM-LIOUVILLE PROBLEM ENERGY DEPENDENT POTENTIAL

In this part, we will give same results for Sturm-Liouville problem energy-dependent potential which is different from the problem given in section 3. Especially, we show that Ambarzumyan theorem is true for this problem.

Let's consider following problem

$$-y''(x) + \{q(x) + 2\lambda p(x)\} y(x) = \lambda^2 y(x) \tag{4.1}$$

$$y'(0) = 0,\ y'(\pi) + f(\lambda) y(\pi) = 0 \tag{4.2}$$

which differs from the usual Sturm-Liouville eigenvalue problem in appearance of the eigenvalue parameter λ in the boundary condition at π.

Throughout this part, we assume that $q, p \in C[0,\pi]$ and

$$f(\lambda) = a_1\lambda + a_2\lambda^2 + a_3\lambda^3 + ... + a_m\lambda^m, a_m \in \mathbb{R}, a_m \neq 0, m \in \mathbb{Z}^+. \tag{4.3}$$

Denote the boundary value problem $(4.1) - (4.3)$ by $B = B(q, p, f(\lambda))$.

Let $\varphi_1(x,\lambda)$ satisfies the differential equation

$$-y''(x) + \{q(x) + 2\lambda p(x)\} y(x) = \lambda^2 y(x) \tag{4.4}$$

and initial conditions

$$y(0) = 1, y'(0) = 0.$$

Then, the solution $\varphi_1(x,\lambda)$ can be expressed as

$$\varphi_1(x,\lambda) = \cos[\lambda x - \alpha(x)] + \int_0^x A(x,t)\cos\lambda t dt + \int_0^x B(x,t)\sin\lambda t dt \tag{4.5}$$

for

$$\alpha(x) = x.p(0) + \int_0^x [A(t,t)\sin\alpha(t) - B(t,t)\cos\alpha(t)]\, dt \tag{4.6}$$

where $A(x,t), B(x,t)$ are continuous with respect to t and x and satisfy the following equation and conditions [15,18,19].

$$\frac{\partial^2 A(x,t)}{\partial x^2} - 2p(x)\frac{\partial B(x,t)}{\partial t} - q(x) A(x,t) = \frac{\partial^2 A(x,t)}{\partial t^2}, \tag{4.7}$$

$$\frac{\partial^2 B(x,t)}{\partial x^2} + 2p(x)\frac{\partial A(x,t)}{\partial t} - q(x) B(x,t) = \frac{\partial^2 B(x,t)}{\partial t^2}, \tag{4.8}$$

$$q(x) = -p^2(x) + 2\frac{d}{dx}\left[A(x,x)\cos\alpha(x) + B(x,x)\sin\alpha(x)\right], \tag{4.9}$$

$$A(0,0) = 0, B(x,0) = 0, \left.\frac{\partial A(x,t)}{\partial t}\right|_{t=0} = 0, \alpha(\pi) = \alpha'(\pi) = 0. \tag{4.10}$$

$$\alpha(x) = \int_0^x p(t)\,dt. \tag{4.11}$$

Similarly, let $\varphi_2(x,\lambda)$ be a solution of the problem

$$-y''(x) + \{q(x) + 2\lambda p(x)\}\,y(x) = \lambda^2 y(x). \tag{4.12}$$

$$y(0) = 0.$$

Then, the solution $\varphi_2(x,\lambda)$ can be expressed as [15]

$$\varphi_2(x,\lambda) = \frac{\sin[\lambda x - \alpha(x)]}{\lambda - \alpha'(x)} + \frac{1}{\lambda}\int_0^x A(x,t)\sin\lambda t dt + \frac{1}{\lambda}\int_0^x B(x,t)\cos\lambda t dt. \tag{4.13}$$

Applying integration by parts to (4.5), we obtain

$$\begin{aligned}
\varphi_1(x,\lambda) &= \cos[\lambda x - \alpha(x)] + \frac{1}{\lambda}A(x,x)\sin\lambda x - \frac{1}{\lambda}\int_0^x \frac{\partial A(x,t)}{\partial t}\sin\lambda t dt \\
&\quad - \frac{1}{\lambda}B(x,x)\cos\lambda x + \frac{1}{\lambda}\int_0^x \frac{\partial B(x,t)}{dt}\cos\lambda t dt \\
\varphi_1(\pi,\lambda) &= \cos[\lambda\pi - \alpha(\pi)] + \frac{1}{\lambda}A(\pi,\pi)\sin\lambda\pi - \frac{1}{\lambda}\int_0^\pi \frac{\partial A(\pi,t)}{\partial t}\sin\lambda t dt \\
&\quad - \frac{1}{\lambda}B(\pi,\pi)\cos\lambda\pi + \frac{1}{\lambda}\int_0^\pi \frac{\partial B(\pi,t)}{dt}\cos\lambda t dt
\end{aligned} \tag{4.14}$$

or asymptotically

$$\varphi_1(\pi,\lambda) = \cos[\lambda\pi - \alpha(\pi)] + \frac{1}{\lambda}A(\pi,\pi)\sin\lambda\pi + O\left(\frac{1}{\lambda}\right) \tag{4.15}$$

and

$$\begin{aligned}
\varphi_1'(\pi,\lambda) &= -[\lambda - \alpha'(\pi)]\sin[\lambda\pi - \alpha(\pi)] + A(\pi,\pi)\cos\lambda\pi + B(\pi,\pi)\sin\lambda\pi \\
&\quad + O\left(\frac{1}{\lambda}\right)
\end{aligned} \tag{4.16}$$

Eigenvalues of the problem (4.1), (4.2) are zeros of $\varphi_1'(\pi,\lambda)+f(\lambda)\varphi_1(\pi,\lambda)=W(\lambda)$. Then inserting (4.15) and (4.16) in there we obtain

$$\begin{aligned} W(\lambda) &= -\left[\lambda-\alpha'(\pi)\right]\sin\left[\lambda\pi-\alpha(\pi)\right]+A(\pi,\pi)\cos\lambda\pi+B(\pi,\pi)\sin\lambda\pi \\ &+f(\lambda)\left\{\cos\left[\lambda\pi-\alpha(\pi)\right]+\frac{1}{\lambda}A(\pi,\pi)\sin\lambda\pi\right\}+O\left(\frac{1}{\lambda}\right). \end{aligned} \tag{4.17}$$

Lemma 4.1: Assume that $\alpha(\pi)=0$.

1) For $m=1$ and $n\to\infty$,

$$\lambda_n=n+\frac{\arctan a_1}{\pi}+\frac{\int_0^\pi\left[q(x)+p^2(x)\right]dx}{2n\pi}+O\left(\frac{1}{n}\right). \tag{4.18}$$

2) For $m=2$ and $n\to\infty$,

$$\lambda_n=\left(n+\frac{1}{2}\right)+\frac{1}{a_2\left(n+\frac{1}{2}\right)\pi}-\frac{\int_0^\pi\left[q(x)+p^2(x)\right]dx}{(2n+1)\pi}+O\left(\frac{1}{n^2}\right). \tag{4.19}$$

3) If $m\geq 3$ and $n\to\infty$,

$$\lambda_n=\left(n+\frac{1}{2}\right)+\frac{1}{a_m\left(n+\frac{1}{2}\right)^{m-1}\pi}-\frac{\int_0^\pi\left[q(x)+p^2(x)\right]dx}{(2n+1)\pi}+O\left(\frac{1}{n^2}\right). \tag{4.20}$$

Proof: From (4.17), we can write λ is an eigenvalue of B if and only if $W(\lambda)=0$. Then,

$$\begin{aligned} W(\lambda) &= -\left[\lambda-\alpha'(\pi)\right]\sin\left[\lambda\pi-\alpha(\pi)\right]+A(\pi,\pi)\cos\lambda\pi+B(\pi,\pi)\sin\lambda\pi \\ &+f(\lambda)\cos\left[\lambda\pi-\alpha(\pi)\right]+\frac{f(\lambda)}{\lambda}A(\pi,\pi)\sin\lambda\pi+O\left(\frac{1}{\lambda}\right) \end{aligned} \tag{4.21}$$

that implies $W(\lambda)=0$. So,

$$\begin{aligned} &-\left[\lambda-\alpha'(\pi)\right]\sin\left[\lambda\pi-\alpha(\pi)\right]+A(\pi,\pi)\cos\lambda\pi+B(\pi,\pi)\sin\lambda\pi \\ &+f(\lambda)\cos\left[\lambda\pi-\alpha(\pi)\right]+\frac{f(\lambda)}{\lambda}A(\pi,\pi)\sin\lambda\pi+O\left(\frac{1}{\lambda}\right)=0. \end{aligned} \tag{4.22}$$

If $\alpha(\pi)=\alpha'(\pi)=0$, then

$$\begin{aligned} &-\lambda\sin\lambda\pi+A(\pi,\pi)\cos\lambda\pi+B(\pi,\pi)\sin\lambda\pi+f(\lambda)\cos\lambda\pi \\ &+\frac{f(\lambda)}{\lambda}A(\pi,\pi)\sin\lambda\pi+O\left(\frac{1}{\lambda}\right)=0 \end{aligned} \tag{4.23}$$

and for $\lambda \cos \lambda\pi \neq 0$.

$$-\tan \lambda\pi + \frac{A(\pi,\pi)}{\lambda} + \frac{B(\pi,\pi)}{\lambda}\tan \lambda\pi + \frac{f(\lambda)}{\lambda} + \frac{f(\lambda)}{\lambda^2}A(\pi,\pi)\tan \lambda\pi + O\left(\frac{1}{\lambda^2}\right) = 0 \tag{4.24}$$

which implies that

$$\tan \lambda\pi = \frac{f(\lambda)}{\lambda} + \frac{A(\pi,\pi)}{\lambda} + \frac{B(\pi,\pi)}{\lambda}\tan \lambda\pi + \frac{f(\lambda)}{\lambda^2}A(\pi,\pi)\tan \lambda\pi + O\left(\frac{1}{\lambda^2}\right). \tag{4.25}$$

1) If $m = 1$, then by (4.25) we have

$$\begin{aligned}
\tan \lambda\pi &= \frac{a_1\lambda}{\lambda} + \frac{A(\pi,\pi)}{\lambda} + \frac{B(\pi,\pi)}{\lambda}\tan \lambda\pi + \frac{a_1\lambda}{\lambda^2}A(\pi,\pi)\tan \lambda\pi + O\left(\frac{1}{\lambda^2}\right)\\
\tan \lambda\pi &= a_1 + \frac{A(\pi,\pi)}{\lambda} + \frac{B(\pi,\pi)}{\lambda}\tan \lambda\pi + \frac{a_1}{\lambda}A(\pi,\pi)\tan \lambda\pi + O\left(\frac{1}{\lambda^2}\right)\\
\tan \lambda\pi - a_1 &= \frac{A(\pi,\pi)}{\lambda}(1 + a_1\tan \lambda\pi) + \frac{B(\pi,\pi)}{\lambda}\tan \lambda\pi + O\left(\frac{1}{\lambda^2}\right)
\end{aligned}$$

$$\tan(\lambda\pi - \arctan a_1) = \frac{\frac{A(\pi,\pi)}{\lambda}(1 + a_1\tan \lambda\pi) + \frac{a_1 B(\pi,\pi)}{\lambda} + O\left(\frac{1}{\lambda^2}\right)}{[1 + a_1\tan \lambda\pi]}.$$

For $\lambda \to \infty$, $\tan \lambda\pi \cong a_1$

$$\tan(\lambda\pi - \arctan a_1) = \left[\frac{1}{1+a_1^2} + O\left(\frac{1}{\lambda}\right)\right]\cdot\left[\frac{A(\pi,\pi)(1+a_1^2)}{\lambda} + \frac{a_1 B(\pi,\pi)}{\lambda} + O\left(\frac{1}{\lambda^2}\right)\right]$$

or

$$\lambda\pi - \arctan a_1 = \arctan\left(\frac{A(\pi,\pi)}{\lambda} + \frac{a_1 B(\pi,\pi)}{(1+a_1^2)\lambda} + O\left(\frac{1}{\lambda^2}\right)\right).$$

$$\lambda\pi = n\pi + \arctan a_1 + \frac{A(\pi,\pi)}{\lambda} + \frac{a_1 B(\pi,\pi)}{(1+a_1^2)\lambda} + O\left(\frac{1}{\lambda^2}\right)$$

$$\lambda = n + \frac{\arctan a_1}{\pi} + \frac{A(\pi,\pi)}{\lambda\pi} + \frac{a_1 B(\pi,\pi)}{(1+a_1^2)\lambda\pi} + O\left(\frac{1}{\lambda^2}\right).$$

$$\lambda_n = n + \frac{\arctan a_1}{\pi} + \frac{A(\pi,\pi)}{n\pi} + \frac{a_1 B(\pi,\pi)}{(1+a_1^2)n\pi} + O\left(\frac{1}{n^2}\right)$$

and by using (4.11), we can obtain easily

$$\lambda_n = n + \frac{\arctan a_1}{\pi} + \frac{\int_0^\pi [q(x) + p^2(x)]\,dx}{2n\pi} + O\left(\frac{1}{n}\right).$$

2) By using (4.23) again

$$-\frac{\lambda}{f(\lambda)}+\frac{A(\pi,\pi)}{f(\lambda)}\cot\lambda\pi+\frac{B(\pi,\pi)}{f(\lambda)}+\cot\lambda\pi+\frac{A(\pi,\pi)}{\lambda}+O\left(\frac{1}{\lambda^2}\right)=0$$

$$-\frac{\lambda}{f(\lambda)}+\left[\frac{A(\pi,\pi)}{f(\lambda)}+1\right]\cot\lambda\pi+\frac{A(\pi,\pi)}{\lambda}+\frac{B(\pi,\pi)}{f(\lambda)}+O\left(\frac{1}{\lambda^2}\right)=0$$

$$\left[\frac{A(\pi,\pi)}{f(\lambda)}+1\right]\cot\lambda\pi=\frac{\lambda}{f(\lambda)}-\frac{A(\pi,\pi)}{\lambda}-\frac{B(\pi,\pi)}{f(\lambda)}+O\left(\frac{1}{\lambda^2}\right)$$

$$\left[\frac{A(\pi,\pi)}{f(\lambda)}+1\right]\cot\lambda\pi=\frac{\lambda-B(\pi,\pi)}{f(\lambda)}-\frac{A(\pi,\pi)}{\lambda}+O\left(\frac{1}{\lambda^2}\right). \tag{4.26}$$

If $m=2$ in (4.26)

$$\left[\frac{A(\pi,\pi)}{a_1\lambda+a_2\lambda^2}+1\right]\cot\lambda\pi=\frac{\lambda-B(\pi,\pi)}{a_1\lambda+a_2\lambda^2}-\frac{A(\pi,\pi)}{\lambda}+O\left(\frac{1}{\lambda^2}\right)$$

$$\left[O\left(\frac{1}{\lambda}\right)+1\right]\cot\lambda\pi=\frac{\lambda\left(1-\frac{B(\pi,\pi)}{\lambda}\right)}{a_2\lambda^2[\frac{a_1}{a_2\lambda}+1]}-\frac{A(\pi,\pi)}{\lambda}+O\left(\frac{1}{\lambda^2}\right)$$

$$\cot\lambda\pi=\frac{1-O\left(\frac{1}{\lambda}\right)}{a_2\lambda[O\left(\frac{1}{\lambda}\right)+1]}-\frac{A(\pi,\pi)}{\lambda}+O\left(\frac{1}{\lambda^2}\right)$$

$$\cot\lambda\pi=\frac{1}{a_2\lambda}-\frac{A(\pi,\pi)}{\lambda}+O\left(\frac{1}{\lambda^2}\right). \tag{4.27}$$

Then by (4.27) we have

$$\lambda\pi=\operatorname{arccot}\left[\frac{1}{a_2\lambda}-\frac{A(\pi,\pi)}{\lambda}+O\left(\frac{1}{\lambda^2}\right)\right]$$

$$\lambda\pi=\left(n+\frac{1}{2}\right)\pi+\frac{1}{a_2\lambda}-\frac{A(\pi,\pi)}{\lambda}+O\left(\frac{1}{\lambda^2}\right)$$

$$\lambda_n=\left(n+\frac{1}{2}\right)+\frac{1}{a_2\pi\lambda}-\frac{A(\pi,\pi)}{\lambda\pi}+O\left(\frac{1}{\lambda^2}\right).$$

For $n\to\infty,\ \lambda_n\cong\left(n+\frac{1}{2}\right)$, then

$$\lambda_n=\left(n+\frac{1}{2}\right)+\frac{1}{a_2\left(n+\frac{1}{2}\right)\pi}-\frac{A(\pi,\pi)}{\left(n+\frac{1}{2}\right)\pi}+O\left(\frac{1}{n^2}\right)$$

$$\lambda_n=\left(n+\frac{1}{2}\right)+\frac{1}{a_2\left(n+\frac{1}{2}\right)\pi}-\frac{\int_0^\pi\left[q(x)+p^2(x)\right]dx}{(2n+1)\pi}+O\left(\frac{1}{n^2}\right).$$

3) For $m \geq 3$, from (4.26) we get

$$\left[\frac{A(\pi,\pi)}{f(\lambda)}+1\right]\cot\lambda\pi=\frac{\lambda-B(\pi,\pi)}{f(\lambda)}-\frac{A(\pi,\pi)}{\lambda}+O\left(\frac{1}{\lambda^2}\right)$$

$$\left[\frac{A(\pi,\pi)}{a_1\lambda+a_2\lambda^2+a_3\lambda^3+...+a_m\lambda^m}+1\right]\cot\lambda\pi=\frac{\lambda-B(\pi,\pi)}{a_1\lambda+a_2\lambda^2+a_3\lambda^3+...+a_m\lambda^m}-\frac{A(\pi,\pi)}{\lambda}+O\left(\frac{1}{\lambda^2}\right)$$

$$\left[O\left(\frac{1}{\lambda}\right)+1\right]\cot\lambda\pi = \frac{\lambda\left(1-\frac{B(\pi,\pi)}{\lambda}\right)}{a_m\lambda^m[\frac{a_1}{a_m\lambda^{m-1}}+\frac{a_2}{a_m\lambda^{m-2}}+\frac{a_3}{a_m\lambda^{m-3}}+...+1]} -\frac{A(\pi,\pi)}{\lambda}+O\left(\frac{1}{\lambda^2}\right)$$

$$\cot\lambda\pi=\frac{1-O\left(\frac{1}{\lambda}\right)}{a_m\lambda^{m-1}[O\left(\frac{1}{\lambda}\right)+1]}-\frac{A(\pi,\pi)}{\lambda}+O\left(\frac{1}{\lambda^2}\right)$$

$$\cot\lambda\pi=\frac{1}{a_m\lambda^{m-1}}-\frac{A(\pi,\pi)}{\lambda}+O\left(\frac{1}{\lambda^2}\right)$$

$$\cot\lambda\pi=\frac{1}{a_m\lambda^{m-1}}-\frac{A(\pi,\pi)}{\lambda}+O\left(\frac{1}{\lambda^2}\right)$$

and we have

$$\lambda\pi=\operatorname{arccot}\left[\frac{1}{a_m\ \lambda^{m-1}}-\frac{A(\pi,\pi)}{\lambda}+O\left(\frac{1}{\lambda^2}\right)\right]$$

$$\lambda\pi=\left(n+\frac{1}{2}\right)\pi+\frac{1}{a_m\ \lambda^{m-1}}-\frac{A(\pi,\pi)}{\lambda}+O\left(\frac{1}{\lambda^2}\right)$$

$$\lambda=\left(n+\frac{1}{2}\right)+\frac{1}{a_m\ \lambda^{m-1}\pi}-\frac{A(\pi,\pi)}{\lambda\pi}+O\left(\frac{1}{\lambda^2}\right).$$

If $n\to\infty$, $\lambda_n\to\left(n+\frac{1}{2}\right)$, we get,

$$\lambda_n=\left(n+\frac{1}{2}\right)+\frac{1}{a_m\ \left(n+\frac{1}{2}\right)^{m-1}\pi}-\frac{A(\pi,\pi)}{\left(n+\frac{1}{2}\right)\pi}+O\left(\frac{1}{n^2}\right)$$

$$\lambda_n = \left(n+\frac{1}{2}\right) + \frac{1}{a_m \ \left(n+\frac{1}{2}\right)^{m-1} \pi} - \frac{\frac{1}{2}\int_0^\pi \left[q(x)+p^2(x)\right]dx}{\left(n+\frac{1}{2}\right)\pi} + O\left(\frac{1}{n^2}\right)$$

$$\lambda_n = \left(n+\frac{1}{2}\right) + \frac{1}{a_m \ \left(n+\frac{1}{2}\right)^{m-1} \pi} - \frac{\int_0^\pi \left[q(x)+p^2(x)\right]dx}{(2n+1)\pi} + O\left(\frac{1}{n^2}\right).$$

This completes the proof.

Let consider a second quadratic Sturm -Liouville problem

$$-y''(x) + \{\widetilde{q}(x) + 2\lambda p(x)\} y(x) = \lambda^2 y(x), \text{ on } [0,\pi] \tag{4.28}$$

$$y'(0) = 0,\ y'(\pi) + f(\lambda) y(\pi) = 0 \tag{4.29}$$

where $\widetilde{q}(x)$ has the same properties of $q(x)$ and $p(x)$, $\widetilde{q}(x) \in C[0,\pi]$.

Theorem 4.2: Assume that $\sigma(p,q) = \sigma(p,\widetilde{q})$ and $\alpha(\pi) = \alpha'(\pi) = 0$, then $\int_0^\pi [q(x) - \widetilde{q}(x)]\,dx = 0$.

Proof : Let's $\lambda_n \in \sigma(p,\widetilde{q}) = \sigma(p,q)$.

1) If $m=1$, $f(\lambda) = a_1\lambda$ then

$$\lambda_n = n + \frac{\arctan a_1}{\pi} + \frac{\int_0^\pi \left[q(x)+p^2(x)\right]dx}{2n\pi} + O\left(\frac{1}{n}\right) \tag{4.30}$$

and

$$\widetilde{\lambda_n} = n + \frac{\arctan a_1}{\pi} + \frac{\int_0^\pi \left[\widetilde{q}(x)+p^2(x)\right]dx}{2n\pi} + O\left(\frac{1}{n}\right). \tag{4.31}$$

Subtracting (4.30) and (4.31)

$$\begin{aligned}\lambda_n - \widetilde{\lambda_n} &= n + \frac{\arctan a_1}{\pi} + \frac{\int_0^\pi \left[q(x)+p^2(x)\right]dx}{2n\pi} + O\left(\frac{1}{n}\right) \\ &\quad - \left\{ n + \frac{\arctan a_1}{\pi} + \frac{\int_0^\pi \left[\widetilde{q}(x)+p^2(x)\right]dx}{2n\pi} + O\left(\frac{1}{n}\right)\right\}\end{aligned}$$

$$\lambda_n - \widetilde{\lambda}_n = \frac{1}{2n\pi}\int_0^\pi \left[q(x) + p^2(x) - \widetilde{q}(x) - p^2(x)\right]dx = 0$$

and finally we obtain $\int_0^\pi [q(x) - \widetilde{q}(x)]\,dx = 0$.

2) If $m \geq 2$, in this case by using (4.20) and $f(\lambda) = a_1\lambda + a_2\lambda^2 + a_3\lambda^3 + ... + a_m\lambda^m$ then

$$\lambda_n = \left(n+\frac{1}{2}\right) + \frac{1}{a_m \ \left(n+\frac{1}{2}\right)^{m-1}\pi} - \frac{\int_0^\pi \left[q(x)+p^2(x)\right]dx}{(2n+1)\pi} + O\left(\frac{1}{n^2}\right) \tag{4.32}$$

and

$$\widetilde{\lambda_n} = \left(n+\frac{1}{2}\right) + \frac{1}{a_m \ \left(n+\frac{1}{2}\right)^{m-1}\pi} - \frac{\int_0^\pi \left[\widetilde{q}(x)+p^2(x)\right]dx}{(2n+1)\pi} + O\left(\frac{1}{n^2}\right). \tag{4.33}$$

Subtracting (4.32) and (4.33) and using some straight for ward computations

$$\begin{aligned}
\lambda_n - \widetilde{\lambda_n} \quad &= \quad \left(n+\frac{1}{2}\right) + \frac{1}{a_m \ \left(n+\frac{1}{2}\right)^{m-1}\pi} - \frac{\int_0^\pi \left[q(x)+p^2(x)\right]dx}{(2n+1)\pi} + O\left(\frac{1}{n^2}\right) \\
&\quad -\left\{\left(n+\frac{1}{2}\right) + \frac{1}{a_m \ \left(n+\frac{1}{2}\right)^{m-1}\pi} - \frac{\int_0^\pi \left[\widetilde{q}(x)+p^2(x)\right]dx}{(2n+1)\pi} + O\left(\frac{1}{n^2}\right)\right\} \ = \ 0,
\end{aligned}$$

$$-\frac{1}{(2n+1)\pi}\int_0^\pi [q(x)+p^2(x)-\widetilde{q}(x)-p^2(x)]dx = 0,$$

or

$$\int_0^\pi \left[q(x)-\widetilde{q}(x)\right]dx = 0.$$

This completes the proof.

Theorem 4.3: Assume that $q \in C[0,\pi]$ and $\sigma(p,q) = \sigma(0,0)$ and $A(\pi,\pi) \neq 0$ then $q = 0$ almost everywhere on $[0,\pi]$.

Proof: From given assumption by Theorem (4.2), we can obatain easily

$$\int_0^\pi q(x)\,dx = 0. \tag{4.34}$$

Let u_1 be the eigenfunction corresponding to the $\lambda = 0$ eigenvalue Then, by (4.1) and (4.2),

$$\begin{gathered}
-u_1''(x) + q(x)\,u = 0, \ \ x \in [0,\pi] \\
u_1'(0) = u_1'(\pi) = 0.
\end{gathered} \tag{4.35}$$

We claim that $u_1(0) \neq 0$ and $u_1(\pi) \neq 0$. Indeed, otherwise we obtain that $u_1(x) = 0$ .which is a contradiction the fact that $u_1(x)$ is an eigenfunction. Let's consider the condition $u_1'(0) = u_1'(\pi) = 0$. Then,

$$\int_0^\pi \frac{u_1''(x)}{u_1(x)} dx = \int_0^\pi q(x)\, dx,$$

$$\left. \frac{u_1'(x)}{u_1(x)} \right|_0^\pi + \int_0^\pi \left(\frac{u_1'(x)}{u_1(x)} \right)^2 dx = \int_0^\pi q(x)\, dx = 0.$$

Then by using the above conditions we obtain

$$\int_0^\pi \left(\frac{u_1'(x)}{u_1(x)} \right)^2 dx = 0.$$

And it means that $u_1' = 0$ or $u_1 = k$. Inserting this in (4.35) ,we obtain that $q = 0$ almost everywhere on $[0, \pi]$.

5. CONCLUSION

In this study, we consider Sturm-Liouville equation energy dependent potential with boundary condition depends on spectral parameter λ. Especially, in the boundary condition, we have a general function as $f(\lambda) = a_1\lambda + a_2\lambda^2 + a_3\lambda^3 + ... + a_m\lambda^m$. We gave asymptotic forms of eigenvalues for all $m \in Z$. Also, by using these results, we showed that when the spectrums are coincide then the potentials q and $\widetilde{q}$ are the same almost everywhere on $[0, \pi]$. Then, we consider these results, it is possible to say that our results are more general then the classical Sturm-Liouville problem [16-18,20-22].

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