**FIRAT UNIVERSITY**

**GRADUATE SCHOOL of NATURAL AND APPLIED SCIENCE**

# GEOMETRIC NULL CURVE FLOWS and INTEGRABLE SYSTEMS in LORENTZ SPACE

## DOCTOR of PHILOSOPHY DISSERTATION

**ZÜHAL KÜÇÜKARSLAN YÜZBAŞI**
**101121206**

**Department : Mathematics**
**Programme : Geometry**

**Supervisor : Prof. Dr. Mehmet BEKTAŞ**
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Zühal Küçükarslan Yüzbaşı
Elazig, 2014

**CONTENTS**

# ÖZET

## LORENTZ UZAYINDA GEOMETRİK NULL EĞRİ AKIŞLARI ve İNTEGRALLENEBİLİR SİSTEMLERİ

Bu tezin amacı genişletilemeyen null eğrilerin akışlarından hareketle Lorentzian flat simetrik uzayında grup-invaryant soliton denklemleri ve onların integrallenebilir yapılarının çalışılmasıdır.

Tezimizde, öncelikle bir sonraki kısımlarda kullanmamız gereken temel tanım ve teoremler göz önüne alındı. İkinci olarak, geometrik null eğri akışlarıve onların integrallenebilir sistemleri cebirsel ve geometrik olmak üzere iki yol ile gosterildi. Tezimizin geri kalan kısmında ise null eğriler için Burger, Airy , Gradient ve Hamiltonian akışları bulundu.



ANAHTAR KELİMELER : Geometrik Akışlar, Null Eğri, Lorentz Uzay, Lie Cebiri.

## SUMMARY

The focus of this dissertation is to study group-invariant soliton equations in Lorentzian flat symmetric space and their integrability structure arising from flows of non-stretching null curves.

In our thesis, first of all, basic definitions and theorems that we need to use in the following chapters were considered. Secondly, geometric null curve flows and it's integrable systems were showed in two ways: one for the algebraic way and another one for the geometric way. The rest of our thesis, the Burger's, Airy, Gradient and Hamiltonian flows were found for null curves.

KEYWORDS : Geometric Flows, Null Curve, Lorentz Space, Lie Algebra.

**LIST of SYMBOLS**

| | |
|---|---|
| $\gamma$ | null curve |
| $e$ | parallel coframe |
| $\omega$ | linear connection |
| $\chi$ | Gauss curvature |
| $e_+$ | the tangent vector of $\gamma$ null curve |
| $\kappa$ | curvature |
| $\boldsymbol{\tau}$ | torsion |
| $D$ | the total exterior derivative operator |
| $R$ | the infinitesimal generator |
| $\mathcal{H}$ | Hamiltonian operator |

## 1 INTRODUCTION

A close relationship with between the theory of integrable soliton equations and differential geometry, particularly as found in the study of geometric curve flows, has appeared in recent years. There have been several articles dealing with the geometric curve flows. For example, in [1] and [2] were showed the bi-Hamiltonian structure of the two known vector generalizations of the mKdV hierarchy of soliton equations is derived in a geometrical fashion from flows of non- stretching curves in Riemannian symmetric spaces $G/SO(n)$. These spaces are exhausted by the Lie groups $G = SO(n+1), SU(n)$. On the other hand, in [3] was studied integrable systems from inelastic curve flows in 2 and 3 dimensional Minkowskian space by using Hasimoto variables.

Geometric curve flows which are said to define a $G$-invariant evolution equation for curves in $M = S^2 \tilde{-} SO(3)/SO(2)$, $H^2 \tilde{-} SL(2,R)/SO(2)$, $R^2 \tilde{-} Euc(2)/SO(2)$ are well known to yield the mKdV hierarchy of scalar soliton equations and its hereditary recursion operator in [4] and [5]. The derivation of these soliton hierarchies utilizes a moving parallel frame, connection 1-form along the curve flows and tied to a zero curvature Maurer–Cartan form on $G$, related to the Klein geometry of the Lie group $G \supset H$, where $H$ is the local frame structure group. The starting aim is to formulate the flow of a non-stretching (inextensible) curve $\gamma(t,x)$ as geometrically. Then, the integrable equations are given by Cartan's structure equations.

In this thesis, we show that the flows of null curve are not to yield the mKdV equation. However, the group-invariant soliton equations and their integrability structure which are derived from studying non-stretching flows of null curves in Lorentzian flat symmetric spaces $G/H = SO\left(n,1\right)/SO\left(n-1,1\right)$ turn out to be Burger's, Hamiltonian, Gradient and Airy flow null curve in $SO\left(n,1\right)/SO\left(n-1,1\right)$.

In second chapter, we give some basic definitions, theorems and notations about differential geometry, Lie group and algebra, gauge transformation and Hamiltonian operator that will be used in the following chapters.

In third chapter, we show how to derive group-invariant soliton equations and their integrability structure from studying non-stretching flows of null curves in Lorentzian flat symmetric spaces $G/H = SO\left(n,1\right)/SO\left(n-1,1\right)$ by using the algebraic way and the geometric way, respectively.

In the last chapter, the geometric flows for null curves are obtained by using different ways. Firstly, we consider scaling and higher symmetry to find two kinds of Burger's flows which are non-linear equations. The Burger's equations have a long history starting with the papers by Burger's in which the Cole-Hopf transformation was introduced. This transformation turns the homogeneous

Burger's equation into the heat equation. Moreover, this equations represent the simplest wave equations combining both dissipative and non linear effects, thus comes out in a great variety of physical applications in [7]. As further results, we use gauge transformation to find three kinds of flows for null curve. First is the Gradient flows which are evolutionary systems driven by an energy that decreases along solutions, as fast as possible. Second is the Airy flows which are the simplest second-order linear differential equation with a turning point where the character of the solutions changes from oscillatory to exponential. The last is the Hamiltonian flows which will be described and given some informations in the following chapters.

## 2 BASIC DEFINITION

In this chapter, we present some definitions and theorems which will play an important role in the discussions that follow in the later chapters. Also, we give some relevant definitions and theorems.

### 2.1 Lorentzian and Riemannian Space

In this subsection, we give all concepts about Riemannian and Lorentzian manifolds that will be used in the current and following chapters.

**Definition 2.1.1** A curve in a manifold $M$ is a smooth mapping $\alpha : I \to M$, where $I$ is an open interval in the real line $R^1$, [6].

**Definition 2.1.2** A $G$-invariant evolution equation for curves in $M$ is said to define a geometric curve flow, [7].

**Definition 2.1.3** A curve $\alpha$ is an integral curve of $X \in \chi(M)$ if whenever $\alpha(t)$ is in the domain of $X$ , then $\alpha\prime(t) = X_{\alpha(t)}$ for all $t \in I$, [8].

**Definition 2.1.4.** Consider the collection of all integral curves $\alpha : I \to M$ of $V$ that start at $p \in M$, that is, for which $\alpha(0) = p$. Let $\alpha_p : I_p \to M$ be an integral curve such that

$$I_p = \cup I_\alpha.$$

Then, we call $\alpha_p$ the maximal integral curve of $V$ starting at $p$, [6].

**Definition 2.1.5.** If $V$ is a vector field, we denote the parametrized maximal integral curve passing through $p$ in $M$ by $\Psi\,(p, t) = \alpha_p\,(t)$ and call $\Psi$ the flow generated by $V$ such that $\Psi : MxR \to M$, [7].

**Lemma 2.1.1.** If $\Psi$ is the flow of a complete vector field, then:

$i)$ $\Psi_0$ is the identity map id of $M$.

$ii)$ $\Psi_s \circ \Psi_t = \Psi_{s+t}$ for all $s, t \in R$ .

$iii)$ Each stage is a diffeomorphism, with $\Psi_t^{-1} = \Psi_{-t}$, [6].

**Definition 2.1.6.** $\eta : V\text{x}V \to R$ is called a symmetric bilinear form provided

$i)$ $\eta(x, y) = \eta(y, x)$,

$ii)$ $\eta(ax + by, z) = a\eta(x, z) + b\eta(y, z)$,

$\eta(x, ay + bz) = a\eta(x, y) + b\eta(x, z)$

for $\forall a, b \in R$ and $\forall x, y, z \in V$, [6].

**Definition 2.1.7.** A symmetric bilinear form $\eta$ on $V$ is

$(1)$ positive [negative] definite provided $u \neq 0$ implies $\eta(u, u) > 0$ $[< 0]$,

$(2)$ positive [negative] semidefinite provided $\eta(u, u) \geq 0$ $[\leq 0]$ for all $u \in V$.

$(3)$ nondegenerate provided $\eta(u, v) = 0$ for all $v \in V$ implies $u = 0$, [6].

**Definition 2.1.8.** A scalar product $\eta$ on a vector space $V$ is a nondegenerate symmetric bilinear form on $V$, [6].

**Definition 2.1.9**.The index $q$ of a symmetric bilinear form $\eta$ on $V$ is the largest integer that is the dimension of a subspace $W \subset V$ on which $\eta|_W$ is negative definite, [6].

**Lemma 2.1.2.** If $W$ is a subspace of a scalar product space $V$, then

$i)$ $dim\ W + dim W^{\perp} = n = dim V$,

$ii)$ $(W^{\perp})^{\perp} = W$,

where

$$W^{\perp} = \{v \in V : \eta(v, w) = 0\ \forall w \in W\},$$

[6].

**Definition 2.1.10**. A metric tensor $\eta$ on a smooth manifold $M$ is a symmetric nondegenerate $(0, 2)$ tensor field on $M$ of constant index, [6].

**Definition 2.1.11**. A semi-Riemannian manifold is a smooth manifold $M$ furnished with a metric tensor $\eta$, [6].

**Definition 2.1.12**. The common value $q$ of index $\eta$, on a semi-Riemannian manifold $M$ is called the index of $M$: $0 \leq q \leq n = dim M$. If $q = 0$, $M$ is a Riemannian manifold; each $g_p$, is then a (positive definite) inner product on $Tp(M)$. If $q = 1$ and $n \geq 2$, $M$ is a Lorentz manifold, [6].

**Definition 2.1.13**. A tangent vector $u$ to $M$ is

$$\begin{array}{cc} \text{spacelike if} & \eta(u,u) > 0 \text{ or } u = 0, \\ \text{null if} & \eta(u,u) = 0 \text{ and } u \neq 0, \\ \text{timelike if} & \eta(u,u) < 0, \end{array}$$

[6].

**Definition 2.1.14**. The set of all null vectors in $T_p(M)$ is called the null cone at $p \in M$, [6].

**Notations 2.1.1** Let $\alpha : I \to M$ be a null curve of a proper semi-Euclidean manifold $M$, where $I \subset R$ is an open set. Denoting $T\alpha$ the tangent bundle of $\alpha$ and

$$T\alpha^{\perp} = \cup_{p \in \alpha(I)} T_p\alpha^{\perp}, T_p\alpha^{\perp} = \{v_p \in T_pM : \langle \xi_p, v_p \rangle = 0\},$$

where $\xi_p$ is null vector tangent to $\alpha$ at $p = \alpha(t)$ , the tangent bundle $T\alpha$ of $\alpha$ is a vector subbundle of $T\alpha^{\perp}$. Considering a complementary vector bundle $S(T\alpha^{\perp})$ to $T\alpha$ in $T\alpha^{\perp}$. We get the direct orthogonal sum

$$T\alpha^{\perp} = T\alpha \perp S(T\alpha^{\perp}),$$

where $S(T\alpha^{\perp})$ is a nondegenerate vector bundle, thus we get

$$TM|_{\alpha} = S(T\alpha^{\perp}) \perp S(T\alpha^{\perp})^{\perp},$$

where $S(T\alpha^{\perp})^{\perp}$ is a complementary orthogonal vector bundle to $S(T\alpha^{\perp})$ in $TM|_{\alpha}$. $ntr(\alpha)$ denote a subbundle of a screen vector bundle $S(T\alpha^{\perp})^{\perp}$ of such that

$$S(T\alpha^{\perp})^{\perp} = T\alpha \oplus ntr(\alpha),$$

[9].

**Definition 2.1.15**. Given a function $f : M \to R$ and a vector field $X$ on $M$ we will use the following ways of writing the directional derivative of $f$ in the direction of

$$\nabla_X f = D_X f = df(X) = X(f),$$

[10].

**Definition 2.1.16.** A connection $\nabla$ on a smooth manifold $M$ is a function $\nabla : \chi(M) \times \chi(M) \to \chi(M)$ such that

$(D1)$ $\nabla_X Y$ is $\mathcal{F}(M)$-linear in $X$,

$(D2)$ $\nabla_X Y$ is $R$-linear in $Y$,

$(D3)$ $\nabla_X (fY) = (Xf)Y + f\nabla_X Y$ for $f \in \mathcal{F}(M)$.

$\nabla_X Y$ is called the covariant derivative of $Y$ with respect to $X$ for the connection $\nabla$, [6].

**Definition 2.1.17.** A Riemannian connection on a Riemannian manifold $M$ is a connection $\nabla$ on $M$ such that

$(D_4)$ $\nabla_X Y - \nabla_Y X = [X, Y]$ (the connection has zero torsion tensor),

$(D_5)$ $Z \langle X, Y \rangle = \langle \nabla_Z X, Y \rangle + \langle X, \nabla_Z Y \rangle$ , [6].

**Proposition 2.1.1.** Let $M$ be a semi-Riemannian manifold. If $V \in \chi(M)$ let $V^*$ be the one-form on $M$ such that

$$V^*(X) = \langle V, X \rangle$$

for all $X \in \chi(M)$. Then the function $V \to V^*$ is an $\mathcal{F}(M)$-linear isomorphism from $\chi(M)$ to $\chi^*(M)$, [6].

**Definition 2.1.18.** Let $M$ be a semi-Riemannian manifold with Levi- Civita connection $\nabla$. The function $R : \chi(M)^3 \to \chi(M)$ given by

$$R_{XY} Z = [\nabla_X, \nabla_Y] Z - \nabla_{[X,Y]} Z$$

is a $(1, 3)$ tensor field on $M$ called the Riemannian curvature tensor of $M$, [6].

**Definition 2.1.19.** The Riemann–Christoffel curvature tensor (of type $(0, 4)$) is the 4-covariant tensor

$$F(x, y, z, w) = \langle R(z, w)y, x \rangle$$

for any $x, y, z, w$ tangent vectors at $p$, [6].

**Theorem 2.1.1.** The following relations are true

$(i)$ $R(x, y)z + R(z, x)y + R(y, z)x = 0$ (first Bianchi identity),

$(ii)$ $\nabla_z R(x, y)w + \nabla_x R(y, z)w + \nabla_y R(z, x)w = 0$ (second Bianchi identity),

$(iii)$ $F(x,y,z,w) = -F(y,x,z,w) = -F(x,y,w,z)$,

$(iv)$ $F(x,y,z,w) = F(z,w,x,y)$, [8].

**Definition 2.1.20.** Let $\Pi$ be a nondegenerate tangent plane to $M$ at $p$. The number

$$K(x,y) = \frac{F(x,y,x,y)}{A\,(x,y)}$$

is independent of the choice of basis $x, y$ for $\Pi$, and is called the sectional curvature $K(\Pi)$, where $A\,(x,y) = \langle x,x\rangle \langle y,y\rangle - \langle x,y\rangle^2$ , [8].

**Definition 2.1.21.** A Riemannian manifold $M$ is said to have constant Riemannian curvature if the sectional curvature of all plane sections is the constant, [6].

**Definition 2.1.22.** A Riemannian manifold $M$ is said flat manifold if the Riemannian curvature is 0 at $\forall p \in M$, [6].

**Proposition 2.1.2.** If $K = 0$ at $p \in M$, then $R = 0$ at $p \in M$, [6].

**Proposition 2.1.3.** The following properties are equivalent:

$i)$ $sec(\pi) = \chi$ for all 2–planes in $T_pM$.

$ii)$ $R(x,y)z = \chi(\langle y,z\rangle\, x - \langle x,z\rangle\, y)$ for any $x, y, z$ in $T_pM$, [10].

**Corollary 2.1.1.** Assume the manifold $M$ has constant Riemannian curvature. Then

$i)$ $\nabla_x R = 0$ along any direction determined by the vector field $X$. That is, the Riemann curvature tensor is parallel.

$ii)$ If $z$ is orthogonal to $x$ and $y$ , then $R(x,y)z = 0$.

$iii)$ If $w$ is orthogonal to $w$ and $y$ , than $F(x,z,x,y) = 0$ for any $z$, [10].

**Definition 2.1.23.** $M$ be a semi-Riemannian manifold with Levi- Civita connection $D$. The function $T : \chi(M) \times \chi(M) \to \chi(M)$ given by

$$T\,(X,Y) = \nabla_X Y - \nabla_Y X - [X,Y]$$

is a $(1,3)$ tensor field on $M$ called the Torsion tensor of $\nabla$, [6].

**Definition 2.1.24.** For a point of $M$, the dual vector space $T_p^*(M)$ of the tangent space $T_p(M)$ is called the space of covectors at $p$. An assignment of a covector at each point $p$ is called a 1 form, [11].

**Definition 2.1.25.** Let $M$ be a semi-Riemannian manifold. Let $U$ be an open set in $M$ and let $\{e_i\}$be $\mathcal{F}(M)$ vector fields that form an orthonormal basis at each point in $U$. Define the connection 1-forms $\omega_i^j$ on $U$ by

$$\nabla_X e_i = \sum_{j=1}^{n} \omega_i^j \left(X\right) e_j,$$

where

$$\omega_i^j : \underset{X}{\chi(M)} \to \underset{\omega_i^j(X)=\omega^j(D_X e_i)}{C^\infty\left(M,R\right)} \;,$$

[11].

**Definition 2.1.26.** The first and second Cartan structure equations are

$$d\omega^i = -\sum_{p=1}^{n} \omega_p^i \Lambda \omega_p + \frac{1}{2} T_{jk}^i \omega^j \Lambda \omega^k,$$
$$d\omega_l^i = -\sum_{j=1}^{n} \omega_p^i \Lambda \omega_p^l + \frac{1}{2} R_{ljk}^i \omega^j \Lambda \omega^k,$$

respectively, where $\omega^i$ is a 1-form and $\omega_i^j$ is a connection 1-form, [11].

**Definition 2.1.27.** A $r$-form $\varphi$ and a vector field $v$, the left-hook $v \lrcorner \varphi$ of $\varphi$ with $v$ (also called the interior product of $\varphi$ with $v$) is the $(r-1)$-form defined by the property that for any $w_1, ..., w_{r-1} \in T_pM$

$$(v \lrcorner \varphi)(w_1, ..., w_{r-1}) = \varphi(v, w_1, ..., w_{r-1}),$$

[11].

**Theorem 2.1.2.** A manifold $M$ is affine locally symmetric if and only if $T = 0$ and $\nabla_z R = 0$ for all $Z \in \mathcal{F}(M)$, [11].

## 2.2 Lie algebra

In this subsection, we review some basic definitions, notations and important theorems about Lie algebra that will be used frequently in the next chapters.

**Definition 2.2.1**. Let $G$ be a group. A subgroup $H$ of $G$ is said to be a normal (or invariant) subgroup of $G$ if $aH = Ha$ for all $a \in G$, [12].

**Definition 2.2.2.** Let $N \trianglelefteq G$. The quotient group or factor group, $G/N$ (read $G$ modulo $N$ or simply $G$ mod $N$) is

$$G/N = \{gN | g \in G\},$$

[12].

**Definition 2.2.3.** A Lie group is a group $G$ which is also an differentiable manifold such that the mapping $(\sigma, \tau) \to \sigma\tau^{-1}$ of the product manifold $GXG$ into $G$ is differentiable, [12].

**Definition 2.2.4.** Let $g$ be a vector space over a field $K$ ( of characteristic $0$). The set $g$ is called a Lie algebra over K if there is given a rule of composition $(X, Y) \to [X, Y]$ in a which is bilinear and satisfies

$i)$ $[X, X] = 0$ for all $X \in g$;

$ii)$ $[X, [Y, Z]] + [Y, [Z, X]] + [Z, [X, Y]] = 0$ for all $X, Y, Z \in g$, [12].

**Definition 2.2.5.** Let $G$ be a Lie group with Lie algebra $g$. A differential form $w$ on $G$ is called left invariant if

$$L(x)^* w = w$$

for all $x \in G$, $L(x)$ denoting the left translation $g \to xg$ on $G$. Similarly we define right invariant differential forms on $G$. A form is called bi-invariant if it is both left and right invariant, [12].

**Definition 2.2.6.** Let $g$ be a Lie algebra over a field of characteristic $0$. Denoting by $Tr$ the trace of a vector space endomorphism we consider the bilinear form

$$B(X, Y) = Tr(adX, adY)$$

on $g \times g$. The form $B$ is called the Killing form of $g$. It is clearly symmetric, [12].

**Definition 2.2.7.** A Lie algebra $g$ over a field of characteristic 0 is called semisimple if the Killing $B$ of $g$ is nondegenerate. We shall call a Lie algebra $g \neq \{0\}$ simple t if it is semisimple and has no ideals except $\{0\}$ and $g$. A Lie group is called semisimple (simple) if its Lie algebra is semisimple (simple), [12].

**Definition 2.2.8.** If $a$ is a Lie algebra over $K$ and $X \in a$, the linear transformation

$$(adX)\,Y \rightarrow [X, Y]$$

of $a$ is denoted by $adX$, [12].

**Definition 2.2.9.** Let $a$ and $b$ be two Lie algebras over the same field $K$ and $\sigma$ a linear mapping of $a$ into $b$.

$(i)$ The mapping $\sigma$ is called a homomorphism if $\sigma([X, Y]) = [\sigma X, \sigma Y]$ for all $X, Y \in \sigma$.

$(ii)$ If $\sigma$ is a homomorphism then $\sigma(a)$ is a subalgebra of $b$ and the kernel $\sigma^{-1}\{0\}$ is an ideal in $a$. If $\sigma^{-1}\{0\} = \{0\}$, then $\sigma$ is called an isomorphism of $a$ into $b$.

$(iii)$ An isomorphism of a Lie algebra onto itself is called an automorphism, [12].

**Definition 2.2.10.** If $a$ is a Lie algebra and $b$, $c$ subsets of $a$,

$(i)$ The centralizer of $b$ in $c$ is

$$\{X \in c : [X, b] = 0\}.$$

$(ii)$ If $b \subset a$ is a subalgebra, its normalizer in $a$ is

$$n = \{X \in a : [X, b] \subset b\},$$

$b$ is an ideal in $n$, [12].

**Definition 2.2.11.** Let $G$ be a Lie group. A submanifold $H$ of $G$ is called a Lie subgroup if

$i)$ $H$ is a subgroup of the (abstract) group $G$;

$ii)$ $H$ is a topological group, [12].

**Definition 2.2.12.** Let $M$ be a Hausdorff space and $G$ a topological group such that to each $g \in G$ is associated a homeomorphism $\rightarrow g.p$ of $M$ onto itself such that

$i)$ $g_1 g_2.p = g_1.(g_2.p)$ for $p \in M$, $g_1, g_2 \in G$;

$ii)$ the mapping $(g, p) \rightarrow g.p$ is a continuous mapping of the product space $G \times M$ onto $M$.

The group $G$ is then called a topological transformation group of $M$. From $i)$ follows that $e.p = p$ for all $p \in M$. If $e$ is the only element of $G$ which leaves each $p \in M$ fixed then $G$ is called effective and is said to act effectively on $M$, [12].

**Definition 2.2.13.** Let $a$ be a Lie algebra over $R$. Let $GL(a)$ as usual denote the group of all nonsingular endomorphisms of $a$. We recall that an endomorphism of a vector space V (in particular of a Lie algebra) simply means $a$ linear mapping of $V$ into itself. The Lie algebra $gl(a)$ of $GL(a)$ consists of the vector space of all endomorphisms of a with the bracket operation

$$[A, B] = AB - BA.$$

The mapping $X \to ad\ X,\ \ X \in a$ is a homomorphism of a onto a subalgebra $ad(a)$ of $gl(a)$. Let $Int(a)$ denote the analytic subgroup of $GL(a)$ whose Lie algebra is $ad(a)$; $Int\ (a)$ is called the adjoint group of $a$, [12].

**Theorem 2.2.1.** Let $G$ be a Lie group with Lie algebra $g$ and $H$ an (abstract) subgroup of $G$. Suppose $H$ is a closed subset of $G$. Then there exists a unique analytic structure on $H$ such that $H$ is a topological Lie subgroup of $G$, [12].

**Remark 2.2.1.** Let $G$ be a Lie group and $H$ a closed subgroup. The group $H$ will always be given the analytic structure. Let $g$ and $h$ denote the Lie algebras of $G$ and $H$, respectively, and let $m$ denote some vector subspace of $g$ such that $g = h + m$ (direct sum) from Theorem 1.2.1. Let $\pi$ be the natural mapping of $G$ onto the space $G/H$ of left cossets $gH$, $g \in G$. As usual we give $G/H$ the natural topology determined. by the requirement that $\pi$ should be continuous and open, [12].

### 2.3 Definition in the Theory of Non-Stretching Curve Flow and Gauge Freedom

In this subsection, we give some basic definitions, notations about the theory of non-stretching curve flows and Gauge freedom that we need to use in the following chapters .

**Remark 2.3.1.** For any Riemannian symmetric space $M = G/H$, the Lie group $G$ has the structure of a principal $H$-bundle over $M$,

$$\pi : G \to G/H.$$

Consider the Maurer-Cartan form $\omega_G$ on $G$, which is a left- invariant $g$-valued 1-form that provides the canonical identification $T_eG = g$, where $e$ is the identity element of $G$. The Maurer-Cartan form satisfies

$$d\omega_G + \frac{1}{2}[\omega_G, \omega_G]_g = 0$$

and hence, viewed geometrically, it is a zero-curvature connection on the bundle $G$. This collective structure is called the Klein geometry of $M = G/H$. It has a useful reformulation locally on $M$ involving the Lie algebra structure of the tangent spaces of $G/H$, [2].

**Definition 2.3.1.** Let

$$\psi : U \to \pi^{-1}(U) \subset G$$

be any local section of this bundle, where $(U, \psi)$ are coordinates for a local trivialization $G \approx U \times H$. The tangent space of the trivialization $U \times H$ at any point $(x, \psi(x))$ is isomorphic to the Lie algebra $g$ of $G$. In particular, note

$$\pi^* : g \simeq m \times h \to m,$$

as vector spaces, with

$$T_x M \simeq_G m = g/h, \; T_{\psi(x)} H \simeq h,$$

and

$$T_{(x,\psi(x))} G \simeq g = m \oplus h.$$

Now, the pullback of the Maurer-Cartan form $\omega_G$ by $\psi$ yields a $g$-valued $1-$form ${}^g\omega$ (locally) on $M$, satisfying the Cartan structure equation

$$D^g\omega + \frac{1}{2}[{}^g\omega, {}^g\omega]_g = 0,$$

where $D$ is the total exterior derivative operator on $M$ and $[\cdot,\cdot]_g$ is understood to denote the Lie algebra bracket in $g$ composed with the wedge product in $T^*_x M$. (Informally, we can view

$${}^g\omega : T_x M \simeq_G m = g/h \to g,$$

as attaching an infinitesimal Klein geometry to $g/h$.) Note a change of the local section $\psi$, described by a smooth function $h : U \to H$ yielding the local section

$$\tilde{\psi} = \psi h,$$

induces a change of ${}^g\omega$ given by

$$ {}^g\tilde{\omega} = Ad(h^{-1}){}^g\omega + h^{-1}Dh. $$

This is called a gauge transformation on ${}^g\omega$ relative to the gauge $\psi$. Introduce

$$ e := \frac{1}{2}({}^g\omega - \sigma({}^g\omega)), $$

$$ \omega := \frac{1}{2}({}^g\omega + \sigma({}^g\omega)), $$

denoting the even and odd parts of ${}^g\omega$, defined using the automorphism $\sigma$ of $g$. The corresponding decomposition under a gauge transformation on ${}^g\omega$ yields

$$ \tilde{e} = Ad(h^{-1})e, $$

$$ \tilde{\omega} = Ad(h^{-1})\omega + h^{-1}Dh, $$

[2].

**Proposition 2.3.1.** On $U \subset M$, the $m$-valued 1-form $e$ represents a linear coframe and the $h$-valued 1-form $\omega$ represents a linear connection, with $H$ as the structure (gauge) group defining a bundle of linear frames. Let the 2-forms

$$ T = De + [\omega, e]g = De, $$

and

$$ R = D\omega + \frac{1}{2}[\omega, \omega]_h = [\mathcal{D}, \mathcal{D}], $$

denote the $m$-valued torsion and $h$-valued curvature associated to the linear connection and coframe, where

$$ \mathcal{D} = D + [\omega, \cdot]_g $$

is the (gauge) covariant exterior derivative, [2].

**Proposition 2.3.2.** The Cartan structure equation decomposes with respect to $\sigma$ to give

$$T = 0,\ R = -\frac{1}{2}\chi[e, e]_m$$

so thus $D$ is torsion-free and has covariantly-constant curvature, [2].

**Definition 2.3.2.** A $m$-valued linear coframe $e$ along a curve $\gamma(x)$ in an Lorentzian symmetric space $M = G/H$ is said to be $H$-parallel if its associated $h$-valued linear connection $\omega_X = \omega \lrcorner X$ in the tangent direction $X = \gamma_x$ of the curve lies in the perp space $c(e_X)_h^{\perp}$ of the Lie subalgebra $c(e_X)_h \subset h$ of the isotropy gauge group $H_X^* \subset H^*$ that preserves $e_X$ given by a fixed unit vector in $a \subset m$, [2].

**Proposition 2.3.3.** By means of a gauge transformation in the isotropy gauge group $H_X^*$, an $H$-parallel frame $\{(e_{\parallel}^*)_a, (e^* \perp)_a\}$ exists for any curve $\gamma(x)$ in a given Lorentzian symmetric space $G/H$ and is unique up to the equivalence group $H_{\parallel}^* \subset H^*$ consisting of rigid (x-independent) gauge transformations that preserve the algebraic structure

$$m = c(e_X)_m \oplus c(e_X)_m^{\perp},$$
$$h = c(e_X)_h \oplus c(e_X)_h^{\perp},$$

$$\omega_X^{\parallel} := \omega \lrcorner X = 0,$$

along with

$$D_x e_X = 0, |e_X|_m = 1.$$

of the framing along $\gamma(x)$ where the dual frame

$$e^* = e_{\parallel}^* + e_{\perp}^*,$$

and the connection

$$\omega = \omega^{\parallel} + \omega^{\perp},$$

[2].

**Lemma 2.3.1.** In an $H$-parallel frame formulation for non-stretching curve flows, the projections of $e$ and $\omega$ in the tangent direction $X = \gamma_x$ and the flow direction $Y = \gamma_t$ have the following properties:

$(i)$ $e_x$ belongs to some Cartan space $a \subset m$, is constant ($D_x e_x = 0$), and has the normalization $|ex|_m = 1$;

$(ii)$ $e_t = h_{\|} + h_{\perp}$ and $\omega_t = \omega^{\|} + \omega^{\perp}$ each decompose into algebraically parallel and perpendicular parts relative to $e_x$, namely $h_{\|}$ an d $\omega^{\|}$ belong to the centralizer subspaces $m_{\|} := c(e_X)_m$ and $h_{\|} := c(e_X)_h$ while $h_{\perp}$ and $\omega^{\perp}$ belong to the perp spaces $m_{\perp} := c(e_X)_m^{\perp}$ and $h_{\perp} := c(e_X)_h^{\perp}$ ;

$(iii)$ $u := \omega_x$ belongs to the perp space $h_{\perp}$;

$(iv)$ $h_{\perp}, e_x, u$ respectively determine the perpendicular flow vector

$$Y_{\perp} = (\gamma_t)_{\perp} = -\langle h_{\perp}, e^* \rangle_m ,$$

the tangent vector

$$X = \gamma_x = -\langle e_x, e^* \rangle_m ,$$

and the principal normal vector

$$N = \nabla_x \gamma_x = -\langle ad\,(e_x)\,u, e^* \rangle_m ,$$

for the curve, [2].

**Definition 2.3.4**. Let $U, V$ be two vector spaces. Any mapping from $U$ to $V$ is called an operator. Let $V$ be a vector space over the field $K$. We can define the structure of a vector space on the set of all operators from $U$ to $V$ ($A$ and $B$ are operators):

$$(A + B)x = Ax + Bx,$$

$$(\alpha A)x = \alpha Ax$$

for all $A, B : U \to V$, for all $x$ in $U$ and for all $\alpha$ in $K$.

**Definition 2.3.5.** Let $A$ be a linear operator on an inner product space $V$. The adjoint of $A$ is a transformation

$$A^* : V^* \to V^*$$

satisfying

$$\langle A\left(x\right), y\rangle = \langle x, A^{*}\left(y\right)\rangle$$

for all $x, y \in V$. Notice that the adjoint of $A$ may not exist. If $A$ is finite-dimensional, then the adjoint operator $A^{*}$ always exist.

**Definition 2.3.6. i**) The operator $A$ is self adjoint if $A^{*} = A$.
$ii$) The operator $A$ is anti-self adjoint if $A^{*} = -A$.

**Notification 2.3.1.** Let $M \subset X \times U$ be an open subset of the space of independent and dependent variables $x = (x_1, \cdots, x_p)$ and $u = (u_1, \cdots, u_q)$. The algebra of differential functions $P[u]$ over $M$ is denoted by $A$, and its quotient space under the image of the total divergence is the space $\mathcal{F}$ of functionals

$$\mathcal{P} = \int P dx. \tag{2.1}$$

For a linear differential operator $\mathcal{D} : \mathcal{A}^q \rightarrow \mathcal{A}^q$, which it can be thought as $q \times q$ matrix differential operator depending on $x, u$ and derivatives of $u$, we define a bracket on $\mathcal{F}$ as follows:

$$\{P, Q\} = \int \delta\mathcal{P}.\mathcal{D}\delta Q dx,$$

where $\delta\mathcal{P}$ is variational derivative of functional $\mathcal{P}$ and where by . is the usual dot product in $R^q$, [13].

**Definition 2.3.7**. A linear operator $\mathcal{D} : \mathcal{A}^q \rightarrow \mathcal{A}^q$ is called Hamiltonian if the bracket (2.1) satisfies the conditions of skew-symmetry

$$\{P, Q\} = -\{Q, P\},$$

and Jacobi identity

$$\{\{P, Q\}, S\} + \{\{S, P\}, Q\} + \{\{Q, S\}, P\} = 0,$$

for all functionals $P, Q, S \in \mathcal{F}$. The bracket $\{P, Q\}$ is called Poisson bracket.

An evolution system is a Hamiltonian system if for a Hamiltonian operator $\mathcal{D}$, there exist a functional $\mathcal{H}$, called Hamiltonian, such that

$$u_t = K[u] = \mathcal{D}\delta\mathcal{H},\ K[u] \in \mathcal{A}^q,$$

[13].

**Definition 2.3.8.** A linear differential operator $\mathcal{R} : \mathcal{A}^q \to \mathcal{A}^q$ is a recursion operator of $u_t = K[u]$ if it maps a symmetry to a new symmetry.

Namely, $\mathcal{R}$ is a recursion operator of $u_t = K[u]$ if and only if

$$\mathcal{R}_t = [D_K, \mathcal{R}],$$

[13].

**Definition 2.3.9.** Consider a system of evolution equations

$$u_t = P[u],$$

in which $P[u] = P\left(x, u^n\right) \in \mathcal{A}^q$ depends on $x \in R^p$, $u \in R^q$ and $x-$derivatives of $u$ only, [7].

## 3 NON-STRETCHING GEOMETRIC FLOWS of NULL CURVE in $SO(n,1)/SO(n-1,1)$

The main goal of this section is to give how to derive group-invariant soliton equations and their integrability structure from studying non-stretching flows of null curves in Lorentzian symmetric spaces $G/H = SO\left(n,1\right)/SO\left(n-1,1\right)$ by using the algebraic way and the geometric way. Then, we get Burger's, Hamiltonian, Gradient and Airy flow for non-stretching null curve in $SO\left(n,1\right)/SO\left(n-1,1\right)$ by choosing transformation on this equations what we obtain from torsion and curvature equations.

### 3.1 Algebraic Way to Calculate Flows of Null Curve in Lorentzian Space

Consider an arbitrary flow $\gamma(t,x)$ of any null curve in an Lorentzian symmetric space $M = G/H$. We can write $X = \gamma_x$ for the tangent vector and $Y = \gamma_t$ for the evolution vector along the null curve. $\gamma(t,x)$ will describe a smooth two-dimensional surface immersed in the manifold $M$ if the flow is transverse to the curve, namely $X$ and $Y$ are not parallel vectors.

Also, provided a null curve flow $\gamma(t,x)$ preserves the arc-length

$$ds = |\gamma_x|dx$$

at every point on the curve, that is,

$$Dt|\gamma_x| = 0$$

or equivalently

$$\nabla_t\gamma_x(= \nabla_x\gamma t)$$

is orthogonal to $\gamma_x$ in the Lorentzian metric $\eta$ on $M$, then it is called non- stretching (inextensible) null curve flow. Now let $\gamma(t,x)$ be a non-stretching null curve flow given in terms of an affine parametrization $|\gamma_x| = 1$, without loss of generality, where $x$ measures arc length along the curve. For any such flow, there exists an $H$-parallel frame formulation obtained through Proposition 2.3.1. In particular, a suitable local gauge transformation in $H^* \simeq Ad(H)$ (given by a change of coordinates for the local trivialization of $G$) can be applied to the linear coframe $e$ and linear connection $\omega$ to produce an $H$-parallel moving frame along each curve in the flow $\gamma(t,x)$ , depending smoothly on $t$.

### 3.1.1 Lorentzian metric and connection with lie algebra

The Lorentzian structure of $G/H$ comes from the Cartan-Killing form $\langle , \rangle$ of $g = so\,(n,1)$ which has indefinite sign. Since

$$g = so\,(n,1) = h \oplus m$$

is a non-compact semi-simple Lie algebra, the subspaces. We consider

$$\mathrm{g} = \begin{pmatrix} 0 & \overrightarrow{g} \\ \overrightarrow{g}^T & A \end{pmatrix},\ A{\in}so(n), \tag{3.1}$$

where $A^T = -A$ and $\overrightarrow{g} = (g_1, ..., g_n) \in R^n$.

$$h = so\,(n-1,1)\,,\ m = so\,(n,1)\,/so\,(n-1,1)$$

are orthogonal

$$\langle h, m \rangle = 0,$$

here

$$dimg = \frac{n(n+1)}{2},\ dimh = \frac{n(n-1)}{2} \text{ and } \dim m = n.$$

Also $h$ and $m$ satisfy the commutator relations

$$[h,h] \subseteq h, [h,m] \subseteq m\ and\ [m,m] \subseteq h.$$

The Lorentzian $n$ -space $m$ is given by indefinite flat metric in terms of Cartan-Killing inner product

$$\eta\,(,) = -dx_1^2 + \sum dx_i^2,$$

where $x_i$ is a rectangular coordinate system of Lorentzian-space.

Let $\gamma$ be a non-stretching null curve which preserves arc-length that is

$$D_t \gamma_x = 0.$$

If

$$\eta\left(\gamma_x, \gamma_x\right) > 0, \eta\left(\gamma_x, \gamma_x\right) < 0 \text{ or } \eta\left(\gamma_x, \gamma_x\right) = 0,$$

respectively, $\gamma$ is spacelike, timelike or null curve which has an affine parameter. If we adapt these curves with respect to Cartan-Killing product we write

$$\langle e_x, e_x \rangle = -1, \langle e_x, e_x \rangle = 1 \text{ or } \langle e_x, e_x \rangle = 0$$

for spacelike, timelike or null curve,respectively.

Under the canonical identification $m \simeq_G T_x M$, the negative-definite inner product $\langle , \rangle_m$ defines a Riemannian metric tensor $\eta$ on $M$ given by

$$\eta\left(X, Y\right) = -\left\langle X, Y \right\rangle_m$$

for all $G-$invariant vector fields $X, Y$ in $T_x M$. Associated to this metric $\eta$ is a unique torsion-free $G$-invariant Riemannian connection $\nabla$, determined by $\nabla \eta = 0$.

The Torsion equation is given by

$$T(X, Y) = \nabla_X Y - \nabla_Y X - [X, Y] = 0 \tag{3.2}$$

is the torsion-free property, where $[\cdot, \cdot]$ is the commutator of vector fields on $M$.

The Riemann curvature tensor of the connection $\nabla$ is given by

$$R(X, Y) = [\nabla_X, \nabla_Y] - \nabla_{[X,Y]} = -adm([X, Y]_m), \tag{3.3}$$

where $adm$ is defined as a linear map on $TxM \simeq_G m$. This curvature tensor is $G$- invariant and covariantly constant $\nabla R = 0$.

### 3.1.2 Geometric Flows for null curves by using algebraic way in Flat Lorentzian Space

Let $\gamma(t,x)$ be a flow of a non-stretching null curve in $G = SO(n,1)/SO(n-1,1)\tilde{\ }H_1^n$. The Lorentzian covariant derivative of a $SO(n-1,1)$-parallel coframe $e$ such that its associated linear connection $\omega \in T_\gamma^* M \otimes so(n-1,1)$ are defined by

$$\nabla_x e = [e, w \lrcorner \gamma_x] \text{ and } \nabla_t e = [e, w \lrcorner \gamma_t]$$

on the surface swept out by the null curve flow $\gamma(t,x)$. Here

$$e_x = e \lrcorner \gamma_x, e_t = e \lrcorner \gamma_t, \tag{3.4}$$

$$\omega_x = \omega \lrcorner \gamma_x, \omega_t = \omega \lrcorner \gamma_t.$$

In frame components the Torsion (3.2) and Curvature (3.3) equations look like

$$D_t e_x + [\omega_t, e_x] = D_x e_t + [\omega_x, e_t], \tag{3.5}$$

$$D_t \omega_x - D_x \omega_t + [\omega_t, \omega_x] = -\chi \left[e_t, e_x\right], \tag{3.6}$$

where $\chi$ is Gauss curvature. The below equation is the frame components in $m$ of the tangent vector $\gamma_x$ of the curve,

$$e_x = \begin{pmatrix} 0 & \begin{pmatrix} 0 & \vec{0} \end{pmatrix} & 1 \\ \begin{pmatrix} 0 \\ \vec{0}^T \end{pmatrix} & \begin{pmatrix} 0 & \vec{0} \\ \vec{0}^T & \mathbf{0} \end{pmatrix} & \begin{pmatrix} 1 \\ \vec{0}^T \end{pmatrix} \\ 1 & -\begin{pmatrix} 1 & \vec{0} \end{pmatrix} & 0 \end{pmatrix} \in \mathrm{m} \tag{3.7}$$

parameterized by the $n+1$ component vector and **0** $\in so(n-2) \subset so(n)$. The Cartan-Killing inner product on $g$ is given by the trace of the product of an $so\,(n,1)$ matrix and a transpose $so\,(n,1)$ matrix. This inner product equals zero for null curves such that

$$\langle e_x, e_x \rangle_m = tr(e_x . e_x) = 0.$$

Now we will decompose spaces $m$ and $h$ with respect to $ad\,(e_x)$ . First we write out matrices for frame components of $h$ and $m$. Next we write

$$\mathrm{h} = \begin{pmatrix} 0 & \begin{pmatrix} h & \overrightarrow{h} \end{pmatrix} & 0 \\ \begin{pmatrix} h \\ \overrightarrow{h}^T \end{pmatrix} & \begin{pmatrix} 0 & \overrightarrow{h_1} \\ \overrightarrow{-h_1}^T & H_1 \end{pmatrix} & \begin{pmatrix} 0 \\ \overrightarrow{0}^T \end{pmatrix} \\ 0 & \begin{pmatrix} 0 & \overrightarrow{0} \end{pmatrix} & 0 \end{pmatrix}, \tag{3.8}$$

where $H_1^T = -H_1$ ($n-2 \times n-2$ components ) and $\overrightarrow{h}, \overrightarrow{h_1} \in R^{n-2}$.

We find

$$\mathrm{m} = \begin{pmatrix} 0 & \begin{pmatrix} 0 & \overrightarrow{0} \end{pmatrix} & m \\ \begin{pmatrix} 0 \\ \overrightarrow{0}^T \end{pmatrix} & \begin{pmatrix} 0 & \overrightarrow{0} \\ \overrightarrow{0}^T & \mathbf{0} \end{pmatrix} & \begin{pmatrix} m_1 \\ \overrightarrow{m_1}^T \end{pmatrix} \\ m & -\begin{pmatrix} m_1 & \overrightarrow{m_1} \end{pmatrix} & 0 \end{pmatrix}, \tag{3.9}$$

such that $g = so\,(n,1) = h \oplus m$ where $\overrightarrow{0} = (0,...,0) \in R^{n-2}$, $\overrightarrow{m_1} \in R^{n-2}$.

Considering the subgroup $H_{\|} \subset H = SO\,(n-1,1)$ such that

$$Ad\left(H_{\|}\right)\gamma_x = \gamma_x$$

is called Little group with gauge freedom in frame.

$$h = h_{\|} \oplus h_{\perp},$$

which is the lie algebra of $H$, satisfies

$$ad\left(\mathrm{h}_{\|}\right) e_x = \left[\mathrm{h}_{\|}, e_x\right] = 0$$

such that $\mathrm{h}_{\perp} \subset h$ is non unique,

$$\dim h_{\perp} = \dim h - \dim h_{\|}$$

and

$$sp\left(\mathrm{h}_{\|}, \mathrm{h}_{\perp}\right) = \mathrm{h}.$$

Also

$$m = m_{\|} \oplus m_{\perp},$$

which is the lie algebra of $M$, satisfies

$$ad\left(\mathrm{m}_{\|}\right) e_x = \left[\mathrm{m}_{\|}, e_x\right] = 0$$

such that $\mathrm{m}_{\perp} \subset h$ ,

$$\dim m_{\perp} = \dim m - \dim m_{\|}.$$

Also

$$sp\left(\mathrm{m}_{\|}, \mathrm{m}_{\perp}\right) = \mathrm{m},$$

where nor$(m_{\perp})$ and nor$(h_{\perp})$ is in $h$. Now we calculate these

1) Since h=$\mathrm{h}_{\|} \oplus \mathrm{h}_{\perp}$ and $\left\langle \mathrm{h}_{\|}, \mathrm{h}_{\perp} \right\rangle_m$ doesn't have to be equal zero because of null case. Considering

$$\left[\mathrm{h}_{\|}, e_x\right] = 0,$$

thus we have

$$\mathrm{h}_{\|} = \begin{pmatrix} 0 & \begin{pmatrix} 0 & \overrightarrow{h_{\|}} \end{pmatrix} & 0 \\ \begin{pmatrix} 0 \\ \overrightarrow{h_{\|}}^T \end{pmatrix} & \begin{pmatrix} 0 & \overrightarrow{h_{\|}} \\ -\overrightarrow{h_{\|}}^T & H_{\|} \end{pmatrix} & \begin{pmatrix} 0 \\ \overrightarrow{0}^T \end{pmatrix} \\ 0 & \begin{pmatrix} 0 & \overrightarrow{0} \end{pmatrix} & 0 \end{pmatrix}. \tag{3.10}$$

Then, we may choose

$$\mathrm{h}_{\perp} = \begin{pmatrix} 0 & \begin{pmatrix} h_{\perp} & \overrightarrow{h_{\perp}} \end{pmatrix} & 0 \\ \begin{pmatrix} h_{\perp} \\ \overrightarrow{h_{\perp}}^T \end{pmatrix} & \begin{pmatrix} 0 & -\overrightarrow{h_{\perp}} \\ \overrightarrow{h_{\perp}}^T & \mathbf{0} \end{pmatrix} & \begin{pmatrix} 0 \\ \overrightarrow{0}^T \end{pmatrix} \\ 0 & \begin{pmatrix} 0 & \overrightarrow{0} \end{pmatrix} & 0 \end{pmatrix} \tag{3.11}$$

such that

$\mathrm{h}_{\|} \oplus \mathrm{h}_{\perp} \in h.$

2) Since m=$\mathrm{m}_{\|}\oplus\mathrm{m}_{\perp}$ and $\left\langle \mathrm{m}_{\|}, \mathrm{m}_{\perp} \right\rangle_m$ doesn't have to be equal zero because of null case. Considering

$\left[\mathrm{m}_{\|}, e_x\right] = 0,$

we get $m = m_1$ and $\overrightarrow{m_1} = 0$. Thus we find

$$\mathrm{m}_{\|} = \begin{pmatrix} 0 & \begin{pmatrix} 0 & \overrightarrow{0} \end{pmatrix} & m_{\|} \\ \begin{pmatrix} 0 \\ \overrightarrow{0}^T \end{pmatrix} & \begin{pmatrix} 0 & \overrightarrow{0} \\ \overrightarrow{0}^T & \mathbf{0} \end{pmatrix} & \begin{pmatrix} m_{\|} \\ \overrightarrow{0}^T \end{pmatrix} \\ m_{\|} & -\begin{pmatrix} m_{\|} & \overrightarrow{0} \end{pmatrix} & 0 \end{pmatrix}, \tag{3.12}$$

since m=$\mathrm{m}_{\|}\oplus\mathrm{m}_{\perp}$, we can write

$$\mathrm{m}_{\perp} = \begin{pmatrix} 0 & \begin{pmatrix} 0 & \overrightarrow{0} \end{pmatrix} & m_{\perp} \\ \begin{pmatrix} 0 \\ \overrightarrow{0}^T \end{pmatrix} & \begin{pmatrix} 0 & \overrightarrow{0} \\ \overrightarrow{0}^T & \mathbf{0} \end{pmatrix} & -\begin{pmatrix} m_{\perp} \\ \overrightarrow{m_{\perp}}^T \end{pmatrix} \\ m_{\perp} & \begin{pmatrix} m_{\perp} & \overrightarrow{m_{\perp}} \end{pmatrix} & 0 \end{pmatrix}. \tag{3.13}$$

Now we introduce $e_t$,$\mathrm{w}_x$ and $\mathrm{w}_t$ in terms of Lemma 1.3.1

$$e_x = e \lrcorner \gamma_x = a = \begin{pmatrix} 0 & \begin{pmatrix} \overrightarrow{0} & 1 \end{pmatrix} \\ \begin{pmatrix} \overrightarrow{0}^T \\ 1 \end{pmatrix} & \begin{pmatrix} \mathbf{0} & \overrightarrow{e}^T \\ \overrightarrow{e} & 0 \end{pmatrix} \end{pmatrix} \in \mathrm{m}_{\|},\ \mathbf{0} \in so\left(n-1\right), \tag{3.14}$$

where $\langle e_x, e_x \rangle = 0$ such that $\langle \overrightarrow{e}, \overrightarrow{e} \rangle_m = 1.$

$$e_t = \begin{pmatrix} 0 & \begin{pmatrix} 0 & \overrightarrow{0} \end{pmatrix} & h_{\parallel} + h_{\perp} \\ \begin{pmatrix} 0 \\ \overrightarrow{0}^T \end{pmatrix} & \begin{pmatrix} 0 & \overrightarrow{0} \\ \overrightarrow{0}^T & \mathbf{0} \end{pmatrix} & \begin{pmatrix} h_{\parallel} - h_{\perp} \\ -\overrightarrow{h_{\perp}}^T \end{pmatrix} \\ h_{\parallel} & \begin{pmatrix} -h_{\parallel} + h_{\perp} & \overrightarrow{h_{\perp}} \end{pmatrix} & 0 \end{pmatrix} \in \text{m}, \tag{3.15}$$

and

$$\omega_t = \begin{pmatrix} 0 & \begin{pmatrix} w & \overrightarrow{w} + \overrightarrow{\theta} \end{pmatrix} & 0 \\ \begin{pmatrix} w \\ (\overrightarrow{w} + \overrightarrow{\theta})^T \end{pmatrix} & \begin{pmatrix} 0 & -\overrightarrow{w} + \overrightarrow{\theta} \\ -(-\overrightarrow{w} + \overrightarrow{\theta})^T & \mathbf{\Theta} \end{pmatrix} & \begin{pmatrix} 0 \\ \overrightarrow{0}^T \end{pmatrix} \\ 0 & \begin{pmatrix} 0 & \overrightarrow{0} \end{pmatrix} & 0 \end{pmatrix} \in \text{h}, \tag{3.16}$$

where $\mathbf{\Theta}^T = -\mathbf{\Theta}$ ($n-2 \times n-2$ matrix components) and $\overrightarrow{w}, \overrightarrow{\theta} \in R^{n-2}$.

$$\omega_x = \begin{pmatrix} 0 & \begin{pmatrix} u & \overrightarrow{u} \end{pmatrix} & 0 \\ \begin{pmatrix} u \\ \overrightarrow{u}^T \end{pmatrix} & \begin{pmatrix} 0 & -\overrightarrow{u} \\ \overrightarrow{u}^T & \mathbf{0} \end{pmatrix} & \begin{pmatrix} 0 \\ \overrightarrow{0}^T \end{pmatrix} \\ 0 & \begin{pmatrix} 0 & \overrightarrow{0} \end{pmatrix} & 0 \end{pmatrix} \in \text{h}_{\perp}, \overrightarrow{u} \in R^{n-2}. \tag{3.17}$$

From the torsion equation (3.5), we get

$$[\omega_t, e_x] = \begin{pmatrix} 0 & \begin{pmatrix} 0 & \overrightarrow{0} \end{pmatrix} & w \\ \begin{pmatrix} 0 \\ \overrightarrow{0}^T \end{pmatrix} & \begin{pmatrix} 0 & \overrightarrow{0} \\ \overrightarrow{0}^T & \mathbf{0} \end{pmatrix} & \begin{pmatrix} w \\ 2\overrightarrow{w}^T \end{pmatrix} \\ w & -\begin{pmatrix} w & 2\overrightarrow{w} \end{pmatrix} & 0 \end{pmatrix}, \tag{3.18}$$

we obtain

$$
[\omega_x, e_t] = \begin{pmatrix} 0 & \begin{pmatrix} 0 & \overrightarrow{0} \end{pmatrix} & u(h_\parallel - h_\perp) - \overrightarrow{u}\overrightarrow{h_\perp}^T \\ \begin{pmatrix} 0 \\ \overrightarrow{0}^T \end{pmatrix} & \begin{pmatrix} 0 & \overrightarrow{0} \\ \overrightarrow{0}^T & \mathbf{0} \end{pmatrix} & \begin{pmatrix} u(h_\parallel + h_\perp) \\ +\overrightarrow{u}\overrightarrow{h_\perp}^T \\ \overrightarrow{u}^T(h_\parallel + h_\perp) \\ +\overrightarrow{u}^T(h_\parallel - h_\perp) \end{pmatrix} \\ u(h_\parallel - h_\perp) - \overrightarrow{u}\overrightarrow{h_\perp}^T & -\begin{pmatrix} u(h_\parallel + h_\perp) & \overrightarrow{u}(h_\parallel + h_\perp) \\ -\overrightarrow{u}^T\overrightarrow{h_\perp} & +\overrightarrow{u}(h_\parallel - h_\perp) \end{pmatrix} & 0 \end{pmatrix}, \tag{3.19}
$$

and

$$
D_x e_t = \begin{pmatrix} 0 & \begin{pmatrix} 0 & \overrightarrow{0} \end{pmatrix} & D_x(h_\parallel + h_\perp) \\ \begin{pmatrix} 0 \\ \overrightarrow{0}^T \end{pmatrix} & \begin{pmatrix} 0 & \overrightarrow{0} \\ \overrightarrow{0}^T & \mathbf{0} \end{pmatrix} & \begin{pmatrix} D_x(h_\parallel - h_\perp) \\ -D_x\overrightarrow{h_\perp}^T \end{pmatrix} \\ D_x h_\parallel & \begin{pmatrix} D_x(-h_\parallel + h_\perp) & D_x\overrightarrow{h_\perp} \end{pmatrix} & 0 \end{pmatrix}. \tag{3.20}
$$

After we substitute (3.18)−(3.20) into equation (3.5) and using $D_t e_x = 0$, the Torsion equation yields

$$
w = D_x(h_\parallel + h_\perp) + u(h_\parallel - h_\perp) - \overrightarrow{u}\overrightarrow{h_\perp}^T, \tag{3.21}
$$

$$
w = D_x(h_\parallel - h_\perp) + u(h_\parallel + h_\perp) + \overrightarrow{u}\overrightarrow{h_\perp}^T. \tag{3.22}
$$

From (3.21) and (3.22), we get

$$
w = D_x h_\parallel + u h_\parallel \tag{3.23}
$$

and

$$
-2\overrightarrow{w} = D_x\overrightarrow{h_\perp} - \overrightarrow{u}(h_\parallel + h_\perp) + \overrightarrow{u}(-h_\parallel + h_\perp), \tag{3.24}
$$

$$
\overrightarrow{w} = -\frac{1}{2}D_x\overrightarrow{h_\perp} + \overrightarrow{u}h_\parallel. \tag{3.25}
$$

From the Curvature equation (3.6), we calculate

$$[\omega_t,\omega_x]=\begin{pmatrix} 0 & \begin{pmatrix} 2\overrightarrow{\theta}.\overrightarrow{u} & u\left(\overrightarrow{w}-\overrightarrow{\theta}\right)-w\overrightarrow{u}-\overrightarrow{u}\mathbf{\Theta} \end{pmatrix} & 0 \\ \begin{pmatrix} 2\overrightarrow{\theta}.\overrightarrow{u} \\ u\left(\overrightarrow{w}-\overrightarrow{\theta}\right)^T-w\overrightarrow{u}^T+\mathbf{\Theta}\overrightarrow{u}^T \end{pmatrix} & \begin{pmatrix} 0 & -u\left(\overrightarrow{w}+\overrightarrow{\theta}\right)+w\overrightarrow{u}+\overrightarrow{u}\mathbf{\Theta} \\ u\left(\overrightarrow{w}+\overrightarrow{\theta}\right)^T-w\overrightarrow{u}^T+\mathbf{\Theta}\overrightarrow{u}^T & 2\overrightarrow{\theta}\Lambda\overrightarrow{u} \end{pmatrix} & \begin{pmatrix} 0 \\ \overrightarrow{0}^T \end{pmatrix} \\ 0 & \begin{pmatrix} 0 & \overrightarrow{0} \end{pmatrix} & 0 \end{pmatrix}, \tag{3.26}$$

where $2\overrightarrow{\theta}\,\overrightarrow{u}^T=2\overrightarrow{\theta}.\overrightarrow{u}$ and

$$\begin{aligned} 2\overrightarrow{\theta}\Lambda\overrightarrow{u} &= (\overrightarrow{w}+\overrightarrow{\theta})^T\overrightarrow{u}-\overrightarrow{u}^T(\overrightarrow{w}+\overrightarrow{\theta})-(\overrightarrow{w}-\overrightarrow{\theta})^T\overrightarrow{u}+\overrightarrow{u}^T(-\overrightarrow{w}+\overrightarrow{\theta}) \\ &= (\overrightarrow{w}+\overrightarrow{\theta})\Lambda\overrightarrow{u}-(\overrightarrow{w}-\overrightarrow{\theta})\Lambda\overrightarrow{u}=\overrightarrow{w}\Lambda\overrightarrow{u}+2\overrightarrow{\theta}\Lambda\overrightarrow{u}-\overrightarrow{w}\Lambda\overrightarrow{u}. \end{aligned}$$

Also, we get

$$[e_t,e_x]=\begin{pmatrix} 0 & \begin{pmatrix} -2h_\perp & -\overrightarrow{h_\perp} \end{pmatrix} & 0 \\ \begin{pmatrix} -2h_\perp \\ -\overrightarrow{h_\perp}^T \end{pmatrix} & \begin{pmatrix} 0 & -\overrightarrow{h_\perp} \\ \overrightarrow{h_\perp}^T & \mathbf{0} \end{pmatrix} & \begin{pmatrix} 0 \\ \overrightarrow{0}^T \end{pmatrix} \\ 0 & \begin{pmatrix} 0 & \overrightarrow{0} \end{pmatrix} & 0 \end{pmatrix}. \tag{3.27}$$

After we substitute (3.26) and (3.27) into equation (3.6)

$$\begin{aligned} &\begin{pmatrix} 0 & \begin{pmatrix} u_t-D_xw & \overrightarrow{u}_t-D_x\left(\overrightarrow{w}+\overrightarrow{\theta}\right) \end{pmatrix} & 0 \\ \begin{pmatrix} u_t-D_xw \\ \overrightarrow{u}_t^T-D_x\left(\overrightarrow{w}+\overrightarrow{\theta}\right)^T \end{pmatrix} & \begin{pmatrix} 0 & -\overrightarrow{u_t}+D_x\left(\overrightarrow{w}-\overrightarrow{\theta}\right) \\ \overrightarrow{u}_t^T-D_x\left(\overrightarrow{w}-\overrightarrow{\theta}\right)^T & \mathbf{0} \end{pmatrix} & \begin{pmatrix} 0 \\ \overrightarrow{0}^T \end{pmatrix} \\ 0 & \begin{pmatrix} 0 & \overrightarrow{0} \end{pmatrix} & 0 \end{pmatrix} \\ &\quad +[\omega_t,\omega_x] \quad = \quad -\chi\,[e_t,e_x]\,, \end{aligned}$$

therefore

$$u_t=D_xw-2\overrightarrow{\theta}.\overrightarrow{u}+2h_\perp\chi, \tag{3.28}$$

$$\overrightarrow{u}_t-D_x\left(\overrightarrow{w}+\overrightarrow{\theta}\right)+u\left(\overrightarrow{w}-\overrightarrow{\theta}\right)-w\overrightarrow{u}-\overrightarrow{u}\mathbf{\Theta}=\chi\overrightarrow{h_\perp}, \tag{3.29}$$

$$-\overrightarrow{u_t}+D_x\left(\overrightarrow{w}-\overrightarrow{\theta}\right)-u\left(\overrightarrow{w}+\overrightarrow{\theta}\right)+w\overrightarrow{u}+\overrightarrow{u}\mathbf{\Theta}=\chi\overrightarrow{h_\perp}. \tag{3.30}$$

$$D_x\mathbf{\Theta} = 2\overrightarrow{\theta}\Lambda\overrightarrow{u}, \tag{3.31}$$

if we add the equation (3.29) to equation (3.30), we get

$$\overrightarrow{u}_t = D_x w + w\overrightarrow{u} - u\overrightarrow{w} + \overrightarrow{u}\mathbf{\Theta}, \tag{3.32}$$

and if we subtract the equation (3.29) from equation (3.30), we find

$$D_x\overrightarrow{\theta} = -u\overrightarrow{\theta} - \chi\overrightarrow{h_\perp}, \tag{3.33}$$

From the torsion equation, we can find

$$w = D_x h_\parallel + u h_\parallel, \tag{3.34}$$

$$\overrightarrow{w} = -\frac{1}{2}D_x\overrightarrow{h_\perp} + \overrightarrow{u}h_\parallel, \tag{3.35}$$

$$Dx h_\perp = u h_\perp + \overrightarrow{u}\overrightarrow{h_\perp}, \tag{3.36}$$

hence, the curvature equation reduces to

$$u_t = D_x w - 2\overrightarrow{\theta}.\overrightarrow{u} + 2h_\perp\chi, \tag{3.37}$$

$$\overrightarrow{u}_t = D_x\overrightarrow{w} + w\overrightarrow{u} - u\overrightarrow{w} + \overrightarrow{u}\mathbf{\Theta}, \tag{3.38}$$

$$D_x\overrightarrow{\theta} = -u\overrightarrow{\theta} - \chi\overrightarrow{h_\perp}, \tag{3.39}$$

$$D_x\mathbf{\Theta} = 2\overrightarrow{\theta}\Lambda\overrightarrow{u}. \tag{3.40}$$

Here $\otimes$ denotes the outer product of pairs of vectors ($1 \times\ n$ row matrices), producing $n \times n$ matrices

$$\overrightarrow{A} \otimes \overrightarrow{B} = \overrightarrow{A}^T\overrightarrow{B},$$

$\Lambda$ denotes the wedge product of pairs of vectors ($1 \times\ n$ row matrices), producing $n \times n$ matrices

$$\overrightarrow{A}\Lambda\overrightarrow{B} = \overrightarrow{A}^T\overrightarrow{B} - \overrightarrow{B}^T\overrightarrow{A}$$

and $\rfloor$ denotes multiplication of $n \times n$ matrices on vectors ($1 \times n$ row matrices),

$$\overrightarrow{A}\rfloor\left(\overrightarrow{B} \otimes \overrightarrow{C}\right) = \left(\overrightarrow{A}.\overrightarrow{B}\right)\overrightarrow{C}$$

which is the transpose of the standard matrix product on column vectors,

$$(\overrightarrow{B} \otimes \overrightarrow{C})\overrightarrow{A} = (\overrightarrow{C} \cdot \overrightarrow{A})\overrightarrow{B}.$$

Furthermore

$$\overrightarrow{A}\rfloor\left(\overrightarrow{B}\wedge\overrightarrow{C}\right) = \overrightarrow{A}\rfloor\left(\overrightarrow{B} \otimes \overrightarrow{C} - \overrightarrow{C} \otimes \overrightarrow{B}\right) = \left(\overrightarrow{A}.\overrightarrow{B}\right)\overrightarrow{C} - \left(\overrightarrow{C}.\overrightarrow{B}\right)\overrightarrow{A}.$$

In our thesis, we consider $\chi = 0$ and $\overrightarrow{\theta} = 0$ from now, thus $SO(n,1)/SO(n-1,1)$ is being the Lorentzian flat space.

**Theorem 3.1.1.** For all null curve flows,

$i)$ $u_t, w_x$ satisfy the system (3.37) and (3.23) in terms of $h_{\|}$ of the curve.

$ii)$ $\overrightarrow{u}_t, \overrightarrow{w}_x$satisfy the system (3.38) and (3.25) in terms of $\overrightarrow{h_{\perp}}, h_{\|}$ of the curve.

### 3.2 Geometric Way to Calculate Flows of Null Curve in Lorentzian Space

In this section we will give the Lorentzian analogue of a Frenet frame for $\gamma(x)$ such that the tangent vector of $\gamma(x)$ is $e_+$ with Gauge freedom

1)

$$\begin{aligned} e_+ \to \lambda e_+ = \tilde{e}_+, \quad \eta\left(\tilde{e}_+, \tilde{e}_+\right) &= \lambda^2 \eta\left(e_+, e_+\right) = 0 \\ e_+ = \gamma_x \to \tilde{e}_+ &= \lambda\gamma_x = \tilde{\gamma}_x \\ \Longrightarrow \lambda\partial_x &= \partial_{\tilde{x}} \\ = \lambda \widetilde{X}' \partial_{\tilde{x}} \Longrightarrow \widetilde{X}' &= \frac{1}{\lambda}. \end{aligned} \tag{3.41}$$

So $e_+$ equivalent to re-parameterization.

2) Up to re-parameterization $e_+$ is unique then we write

$$e_+ \to \tilde{e}_+ = e_+$$

$$e_- \to \tilde{e}_-$$

$$e_{i\perp} \to \tilde{e}_{i\perp},$$

such that

$$\eta\left(\tilde{e}_-, \tilde{e}_+\right) = -1, \eta\left(\tilde{e}_-, \tilde{e}_-\right) = \eta\left(\tilde{e}_{i\perp}, \tilde{e}_+\right) = \eta\left(\tilde{e}_{i\perp}, \tilde{e}_-\right) = 0 \; and \; \eta\left(\tilde{e}_{i\perp}, \tilde{e}_{j\perp}\right) = \delta_{ij.} \tag{3.42}$$

We would like to calculate

$$D_x \tilde{E} = \widetilde{K} \widetilde{E}, \tag{3.43}$$

in terms of the re-parameterization. Here

$$\widetilde{K} = D_x H H^{-1} + H K H^{-1} \tag{3.44}$$

and $H = \exp(R)$, $R$ is infinitesimal generator such that

$$\mathrm{Re}_+ = R\tilde{e}_+ = 0, \tag{3.45}$$

$$R\tilde{e}_- = ae_+ + b\tilde{e}_- + c^i \tilde{e}_{i\perp}, \tag{3.46}$$

$$R\tilde{e}_{i\perp} = \alpha_i e_+ + \beta_i \tilde{e}_- + \phi_{ij}\tilde{e}_{j\perp}. \tag{3.47}$$

We can write that as  matrix

$$R\begin{pmatrix} e_+ \\ \tilde{e}_- \\ \tilde{e}_{i\perp} \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 \\ a & b & c^j \\ \alpha_i & \beta_i & \phi_{ij} \end{pmatrix} \begin{pmatrix} e_+ \\ \tilde{e}_- \\ \tilde{e}_{j\perp} \end{pmatrix}. \tag{3.48}$$

Now we calculate $R$ operator with respect to the re-parameterization Frenet frame. To get $R$, considering (3.45) and using the re-parameterization Frenet conditions (3.42), we obtain

$$\begin{aligned} \eta\left(\tilde{e}_-, R\tilde{e}_-\right) &= 0 \\ \eta\left(\tilde{e}_-, ae_+ + b\tilde{e}_- + c^j\tilde{e}_{j\perp}\right) &= 0 \\ a\eta\left(\tilde{e}_-, e_+\right) + b\eta\left(\tilde{e}_-, b\tilde{e}_-\right) + c^j\eta\left(\tilde{e}_-, \tilde{e}_{j\perp}\right) &= 0 \\ -a = 0 \Longrightarrow a &= 0, \end{aligned} \tag{3.49}$$

and

$$\begin{aligned} \eta\left(\tilde{e}_-, \mathrm{Re}_+\right) + \eta\left(R\tilde{e}_-, e_+\right) &=0 \Longrightarrow \eta\left(R\tilde{e}_-, e_+\right) = 0 \\ \eta\left(e_+, R\tilde{e}_-\right) &=0 \\ \eta\left(e_+, ae_+ + b\tilde{e}_- + c^j\tilde{e}_{j\perp}\right) &=0 \\ a\eta\left(e_+, e_+\right) + b\eta\left(e_+, b\tilde{e}_-\right) + c^j\eta\left(e_+, \tilde{e}_{j\perp}\right) &=0 \\ -b &=0 \Longrightarrow b = 0. \end{aligned} \tag{3.50}$$

Considering (3.45) and using the re-parameterization Frenet conditions (3.42), we get

$$\begin{aligned} \eta\left(\tilde{e}_{i\perp}, \mathrm{Re}_+\right) + \eta\left(R\tilde{e}_{i\perp}, e_+\right) &= 0 \\ \eta\left(R\tilde{e}_{i\perp}, e_+\right) &= 0 \\ \eta\left(e_+, \alpha_i e_+ + \beta_i\tilde{e}_- + \phi_{ij}\tilde{e}_{j\perp}\right) &= 0 \\ \alpha_i\eta\left(e_+, e_+\right) + \beta_i\eta\left(e_+, \tilde{e}_-\right) + \phi_{ij}\eta\left(e_+, \tilde{e}_{j\perp}\right) &= 0 \\ -\beta_i = 0 \Longrightarrow \beta_i &= 0. \end{aligned} \tag{3.51}$$

and

$$
\begin{aligned}
0 &= \eta\left(R\tilde{e}_-, \tilde{e}_{i\perp}\right) + \eta\left(\tilde{e}_-, R\tilde{e}_{i\perp}\right) \\
0 &= \eta\left(ae_+ + b\tilde{e}_- + c^j\tilde{e}_{j\perp}, \tilde{e}_{i\perp}\right) + \eta\left(\tilde{e}_-, \alpha_i e_+ + \beta_i\tilde{e}_- + \phi_{ij}\tilde{e}_{j\perp}\right) \\
0 &= a\eta\left(e_+, \tilde{e}_{i\perp}\right) + b\eta\left(\tilde{e}_-, \tilde{e}_{i\perp}\right) + c^j\eta\left(\tilde{e}_{j\perp}, \tilde{e}_{i\perp}\right) + \\
&+\alpha_i\eta\left(\tilde{e}_-, e_+\right) + \beta_i\eta\left(\tilde{e}_-, \tilde{e}_-\right) + \phi_{ij}\eta\left(\tilde{e}_-, \tilde{e}_{j\perp}\right) \\
c^j - \alpha_i &= 0 \Longrightarrow c^j = \alpha_i.
\end{aligned}
\tag{3.52}
$$

Finally, we get

$$
\begin{aligned}
0 &= \eta\left(R\tilde{e}_{i\perp}, \tilde{e}_{j\perp}\right) + \eta\left(\tilde{e}_{i\perp}, R\tilde{e}_{j\perp}\right) \\
0 &= \eta\left(\alpha_i e_+ + \beta_i\tilde{e}_- + \phi_{ik}\tilde{e}_{k\perp}, \tilde{e}_{j\perp}\right) + \eta\left(\tilde{e}_{i\perp}, \alpha_j e_+ + \beta_j\tilde{e}_- + \phi_{jk}\tilde{e}_{k\perp}\right) \\
0 &= \alpha_i\eta\left(e_+, \tilde{e}_{j\perp}\right) + \beta_i\eta\left(\tilde{e}_-, \tilde{e}_{j\perp}\right) + \phi_{ik}\eta\left(\tilde{e}_{k\perp}, \tilde{e}_{j\perp}\right) + \\
&\alpha_j\eta\left(\tilde{e}_{i\perp}, e_+\right) + \beta_j\eta\left(\tilde{e}_{i\perp}, \tilde{e}_-\right) + \phi_{jk}\eta\left(\tilde{e}_{i\perp}, \tilde{e}_{k\perp}\right) \\
\phi_{ij} + \phi_{ji} &= 0 \Longrightarrow \phi_{ij} = -\phi_{ji}.
\end{aligned}
\tag{3.53}
$$

Thus, we write the operator $R$ as follows

$$
R = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & \alpha^j \\ \alpha_i & 0 & \phi_{ij} \end{pmatrix} \in so\left(n, 1\right). \tag{3.54}
$$

Considering the above equations, $H$ is given in terms of $R$ as follows

$$
\exp\left(R\right) = 1 + R + \frac{R^2}{2!} + ... + \frac{R^n}{n!} + ... = H\left(\alpha_i, \phi_{ij}\right). \tag{3.55}
$$

Now we calculate $\exp\left(R\right)$. First of all, we get

$$
R^2 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & \alpha^j \\ \alpha_i & 0 & \phi_{ij} \end{pmatrix} \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & \alpha^k \\ \alpha_j & 0 & \phi_{jk} \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 \\ \alpha^j\alpha_j & 0 & \alpha^j\phi_{jk} \\ \phi_{ij}\alpha_j & 0 & \phi_{ij}\phi^j_k \end{pmatrix},
$$

and

$$R^3 = \begin{pmatrix} 0 & 0 & 0 \\ \alpha^j\alpha_j & 0 & \alpha^j\phi_{jk} \\ \phi_{ij}\alpha_j & 0 & \phi_{ij}\phi^j_k \end{pmatrix} \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & \alpha^l \\ \alpha_k & 0 & \phi_{kl} \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 \\ \alpha^j\phi_{jk}\alpha_k & 0 & \alpha^j\phi_{jk}\phi^k_l \\ \phi_{ij}\phi^j_k\alpha^k & 0 & \phi_{ij}\phi^{jk}\phi_{kl} \end{pmatrix},$$

where $\phi_{jk}$ is anti-symmetric, $\alpha^j\phi_{jk}\alpha_k = 0$, therefore

$$R^3 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & \alpha^j\phi_{jk}\phi^{kl} \\ \phi_{ij}\phi^{jk}\alpha_k & 0 & \phi_{ij}\phi^{jk}\phi_{kl} \end{pmatrix},$$

then

$$R^4 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & \alpha^j\phi_{jk}\phi^{kl} \\ \phi_{ij}\phi^{jk}\alpha_k & 0 & \phi_{ij}\phi^{jk}\phi_{kl} \end{pmatrix} \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & \alpha^m \\ \alpha_l & 0 & \phi_{lm} \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 \\ \alpha^j\phi^j_{jk}\phi^{kl}\alpha_l & 0 & \alpha^j\phi_{jk}\phi^{kl}\phi_{lm} \\ \phi_{ij}\phi^{jk}\phi_{kl}\alpha^l & 0 & \phi_{ij}\phi^{jk}\phi_{kl}\phi^{lm} \end{pmatrix}.$$

If we continue these calculations, then we get

$$\exp(R) = \begin{pmatrix} 0 & 0 & 0 \\ A & 0 & B^j \\ C_i & 0 & D_{ij} \end{pmatrix}, \tag{3.56}$$

where

$$\begin{aligned} A &= 0 + \frac{1}{2!}\alpha_j\alpha^j + 0 + \frac{1}{4!}\left(\alpha_j\phi^{jk}\right)\left(-\alpha_l\phi^{lk}\right) + 0 + ... \\ B^j &= 0 + \alpha^j + \frac{1}{2!}\alpha^k\phi_{kj} + \frac{1}{3!}\alpha^l\phi_{lk}\phi^{kj} + ... \\ C_i &= 0 + \alpha_i + \frac{1}{2!}\phi_{ij}\alpha^j + \frac{1}{3!}\phi_{ij}\phi^{jk}\alpha_k + ... \\ D_{ij} &= \delta_{ij} + \phi_{ij} + \frac{1}{2!}\left(\phi_{ik}\phi^{kj}\right) + \frac{1}{3!}\left(\phi_{ik}\phi^{kl}\phi_{lj}\right) + ... \end{aligned} \tag{3.57}$$

Here, if we consider

$$\delta_{ij} \to 1, \phi_{ij} \to \underline{\phi} = -\underline{\phi}^T, \alpha_i \to \vec{\alpha}.$$

The equations (3.57) are rearranged by

$$D_{ij} \to \underline{D} = \Gamma = 1 + \underline{\phi} + \frac{1}{2!}\underline{\phi}^2 + \frac{1}{3!}\underline{\phi}^3 + ... = e^{\underline{\phi}}$$
$$B^j \to \vec{B} = \vec{\alpha} + \frac{1}{2!}\vec{\alpha}\underline{\phi} + \frac{1}{3!}\vec{\alpha}\underline{\phi}^2 + ... = \vec{\alpha}\left(1 + \frac{1}{2!}\underline{\phi} + \frac{1}{3!}\underline{\phi}^2 + ...\right) = \vec{\alpha} f\left(\underline{\phi}\right)$$
$$C_i \to \vec{C} = \vec{\alpha} + \frac{1}{2!}\underline{\phi}\vec{\alpha} + \frac{1}{3!}\underline{\phi}^2\vec{\alpha} + ... = \left(1 + \frac{1}{2!}\underline{\phi} + \frac{1}{3!}\underline{\phi}^2 + ...\right)\vec{\alpha} = f\left(\underline{\phi}\right)\vec{\alpha}.$$

Using $\underline{\phi} f + 1 = e^{\underline{\phi}}$

$$A = \frac{1}{2!}\vec{\alpha}\vec{\alpha} - \frac{1}{4!}\left(\underline{\phi}\vec{\alpha}\right)\left(\underline{\phi}\vec{\alpha}\right) + ... = \frac{1}{2}\left(\vec{\alpha}^T f\right)\left(f^T\vec{\alpha}\right)$$

Also

$$A\left(-\vec{\alpha}, -\underline{\phi}\right) = A\left(\vec{\alpha}, \underline{\phi}\right) = A = \frac{1}{2}\vec{\alpha}^T f\left(\underline{\phi}\right) f\left(\underline{\phi}\right)^T \vec{\alpha}$$
$$\overrightarrow{C}\left(-\vec{\alpha}, -\underline{\phi}\right) \neq \overrightarrow{C}\left(\vec{\alpha}, \underline{\phi}\right) = -f\left(\underline{\phi}\right)^T \vec{\alpha} = f\left(\underline{\phi}\right)\vec{\alpha}$$
$$\underline{D}\left(-\underline{\phi}\right) = \underline{D}^{-1}\left(\underline{\phi}\right) = \underline{D}^T\left(\underline{\phi}\right).$$

where $\vec{\alpha}$ is column vector and $\underline{\phi}$ is a matrix. We get

$$H = \begin{pmatrix} 1 & 0 & 0 \\ A & 1 & \vec{\alpha}^T f \\ f\vec{\alpha} & 0 & \Gamma \end{pmatrix}. \tag{3.58}$$

We can define the Frenet equations for this space. We have

$$\eta\left(e_{+x}, e_{+}\right) = \frac{1}{2}D_x\eta\left(e_{+}, e_{+}\right) = 0 \qquad \Longrightarrow e_{+x} = \mu e_{+} + \kappa^i e_{i\perp}.$$

To get $e_{-x}$, we expand

$$e_{-x} = a_1 e_{+} + a_2 e_{-} + a_3^i e_{i\perp}$$

with coefficients $a_1, a_2$ and $n-2$ components vector $a_3^i$ . By using the Frenet conditions (3.42), we get

$$
\begin{aligned}
\eta\left(e_{-x}, e_{+}\right) &= D_x \eta\left(e_{-}, e_{+}\right) - \eta\left(e_{-}, e_{+x}\right) \\
&= -\eta\left(e_{-}, \mu e_{+} + \kappa^i e_{i\perp}\right) \\
&= -\mu \eta\left(e_{-}, e_{+}\right) - \kappa^i \eta\left(e_{-}, e_{i\perp}\right) \\
&= \mu,
\end{aligned}
$$

and

$$
\eta\left(e_{-x}, e_{-}\right) = \frac{1}{2} D_x \eta\left(e_{-}, e_{-}\right) = 0,
$$

we get $a_1 = -\mu$, $a_2 = 0$, $a_3^i = \tau^i$. Then we can write

$$
\Longrightarrow e_{-x} = -\mu e_{-} + \tau^i e_{i\perp}. \tag{3.59}
$$

To obtain $e_{i\perp x}$, we expand

$$
e_{i\perp x} = b_{1i} e_{+} + b_{2i} e_{-} + b_{ij} e_{j\perp}
$$

with $n-2$ vectors $b_{1i}, b_{2i}$ and $n-2 \times n-2$ skew-symmetric matrix $b_{ij}$ . By applying the Frenet conditions (3.42), we obtain

$$
\begin{aligned}
\eta\left(e_{i\perp x}, e_{+}\right) &= D_x \eta\left(e_{i\perp}, e_{+}\right) - \eta\left(e_{i\perp}, e_{+x}\right) \\
&= 0 - \eta\left(e_{i\perp}, \mu e_{+} + \kappa^i e_{i\perp}\right) \\
&= -\mu \eta\left(e_{i\perp}, e_{+}\right) - \kappa^j \eta\left(e_{i\perp}, e_{j\perp}\right) \\
&= -\kappa_i,
\end{aligned}
$$

and

$$\begin{aligned}
\eta\left(e_{i\perp x}, e_-\right) &= D_x \eta\left(e_{i\perp}, e_-\right) - \eta\left(e_{i\perp}, e_{-x}\right) \\
&= 0 - \eta\left(e_{i\perp}, -\mu e_- + \tau^i e_{i\perp}\right) \\
&= \mu \eta\left(e_{i\perp}, e_-\right) - \tau^j \eta\left(e_{i\perp}, e_{j\perp}\right) \\
&= -\tau_i,
\end{aligned}$$

$$\eta\left(e_{i\perp x}, e_{j\perp}\right) = -\eta\left(e_{i\perp}, e_{j\perp x}\right).$$

thus, we get $b_{1i} = \tau_i$, $b_{2i} = \kappa_i$, $b_{ij} = \sigma_{ij}$. We can write

$$e_{i\perp x} = \tau_i e_+ + \kappa_i e_- + \sigma_{ij} e_{j\perp}, \quad \sigma_{ij} = -\sigma_{ji}. \tag{3.60}$$

we show $K$ as a matrix

$$K = \begin{pmatrix} \mu & 0 & \kappa^T \\ 0 & -\mu & \tau^T \\ \tau & \kappa & \sigma \end{pmatrix}, \tag{3.61}$$

$\mu$ is scalar, $\kappa, \tau$ are vectors, $\sigma$ is antisymmetric matrix.

**Proposition 3.2.1.** For the null curve $\gamma$,

$i)$ $H$ is given by (3.58) in terms of the vector $\vec{\alpha}$ and the anti-symmetric matrix $\underline{\phi}$.

$ii)$ $K$ is given by (3.61) in terms of the vectors $\kappa, \tau$, the scalar $\mu$ and the antisymmetric matrix $\sigma$.

### 3.2.1 Gauge Transformation

Frenet frame using gauge transformation is defined by

$$E = \begin{pmatrix} e_+ \\ e_- \\ e_{i\perp} \end{pmatrix}, \quad \tilde{E} = \begin{pmatrix} e_+ \\ \tilde{e}_- \\ \tilde{e}_{i\perp} \end{pmatrix}. \tag{3.62}$$

Therefore, we want the $x$-derivative of $H$ to have the form $D_x\tilde{E} = \widetilde{K}\tilde{E}$,

$$H = \begin{pmatrix} 1 & 0 & 0 \\ A & 1 & \vec{\alpha}^T f \\ f\vec{\alpha} & 0 & \Gamma \end{pmatrix}, \quad H^{-1} = \begin{pmatrix} 1 & 0 & 0 \\ A & 1 & -\vec{\alpha}^T f^T \\ -f^T\vec{\alpha} & 0 & \Gamma^T \end{pmatrix}, \tag{3.63}$$

$$D_x H = \begin{pmatrix} 0 & 0 & 0 \\ D_x A & 0 & \vec{\alpha}_x^T f + \vec{\alpha}^T D_x f \\ D_x f\vec{\alpha} + f\vec{\alpha}_x & 0 & D_x\Gamma \end{pmatrix}, \tag{3.64}$$

and $A\left(\vec{\alpha}, \underline{\phi}\right)$ is scalar, $f\left(\underline{\phi}\right)$ and $\Gamma\left(\underline{\phi}\right)$ are matrices.

Also $H\left(\vec{\alpha}, \underline{\phi}\right), H^{-1} = H^{-1}\left(-\vec{\alpha}, -\underline{\phi}\right)$

$$\begin{aligned} HH^{-1} &= \begin{pmatrix} 1 & 0 & 0 \\ A & 1 & \vec{\alpha}^T f \\ f\vec{\alpha} & 0 & \Gamma \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ A & 1 & -\vec{\alpha}^T f^T \\ -f^T\vec{\alpha} & 0 & \Gamma^T \end{pmatrix} \\ &= \begin{pmatrix} 1 & 0 & 0 \\ 2A - \vec{\alpha}^T f f^T \vec{\alpha} & 1 & -\vec{\alpha}^T f^T + \vec{\alpha}^T f \Gamma^T \\ f\vec{\alpha} - \Gamma f^T \vec{\alpha} & 0 & \Gamma\Gamma^T \end{pmatrix}. \end{aligned} \tag{3.65}$$

From the above equations, considering $2A - \vec{\alpha}^T f f^T \vec{\alpha} = 0$, $\Gamma^T = \Gamma^{-1}$ and $\Gamma\Gamma^T = 1$, we get

$$\begin{aligned} \left(f - \Gamma f^T\right)\vec{\alpha} &= \left(f - e^{\underline{\phi}} f^T\right)\vec{\alpha} \\ &= \left(f - \left(1 + \underline{\phi} f\right) f^T\right)\vec{\alpha} \\ &= \left(f - f^T - \underline{\phi} f f^T\right)\vec{\alpha}. \end{aligned} \tag{3.66}$$

Taking

$$f = 1 + \frac{1}{2!}\underline{\phi} + \frac{1}{3!}\underline{\phi}^2 + ..., \quad f^T = 1 - \frac{1}{2!}\underline{\phi} + \frac{1}{3!}\underline{\phi}^2 + ...,$$

we obtain

$$
\begin{aligned}
ff^T &= 1 - \frac{1}{2!}\underline{\phi} + \frac{1}{3!}\underline{\phi}^2 + ... \\
&+ \frac{1}{2!}\underline{\phi} - \frac{1}{(2!)^2}\underline{\phi}^2 + ... \\
&+ \frac{1}{3!}\underline{\phi}^2 + ... \\
&= 1 + \frac{1}{2!}\underline{\phi}^2 + ...
\end{aligned} \tag{3.67}
$$

and

$$
\begin{aligned}
f - f^T &= 2\left(\frac{1}{2!}\underline{\phi} + \frac{1}{4!}\underline{\phi}^3 + ...\right) \\
&= 2\underline{\phi}\left(\frac{1}{2!} + \frac{1}{4!}\underline{\phi}^2 + ...\right) \\
&= \underline{\phi} f f^T.
\end{aligned} \tag{3.68}
$$

So we can find

$$
\left(f - \Gamma f^T\right)\vec{\alpha} = \left(f - f^T + \underline{\phi} f f^T\right)\vec{\alpha} = 0 \tag{3.69}
$$

therefore we can easily show

$$
HH^{-1} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix},
$$

$$
D_x E = KE, \;\; K = \begin{pmatrix} \mu & 0 & \kappa^T \\ 0 & -\mu & \tau^T \\ \tau & \kappa & \sigma \end{pmatrix}, \;\; \sigma = -\sigma^T \tag{3.70}
$$

where $\mu$ is scalar, $\kappa, \tau$ are vectors, $\sigma$ is matrix.

Now we calculate the above equation by using equations (3.63) and (3.64), we find

$$D_xHH^{-1} = \begin{pmatrix} 0 & 0 & 0 \\ D_xA & 0 & \vec{\alpha}_x^T f + \vec{\alpha}^T D_x f \\ D_x f\vec{\alpha} + f\vec{\alpha}_x & 0 & D_x\Gamma \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ A & 1 & -\vec{\alpha}^T f^T \\ -f^T\vec{\alpha} & 0 & \Gamma^T \end{pmatrix}$$

$$= \begin{pmatrix} 0 & 0 & 0 \\ D_xA - \left(\vec{\alpha}_x^T f + \vec{\alpha}^T D_x f\right) f^T\vec{\alpha} & 0 & \left(\vec{\alpha}_x^T f + \vec{\alpha}^T D_x f\right)\Gamma^T \\ D_x f\vec{\alpha} + f\vec{\alpha}_x - D_x\Gamma f^T\vec{\alpha} & 0 & D_x\Gamma\Gamma^T \end{pmatrix}, \tag{3.71}$$

we obtain

$$HKH^{-1} = \begin{pmatrix} \mu - \kappa^T f^T\vec{\alpha} & 0 & \kappa^T\Gamma^T \\ \begin{matrix} A\mu + \vec{\alpha}^T f\tau + \left(-\mu + \vec{\alpha}^T f\kappa\right) A \\ - \left(A\kappa^T + \tau^T + \vec{\alpha}^T f\sigma\right) f^T\vec{\alpha} \end{matrix} & -\mu + \vec{\alpha}^T f\kappa & \begin{matrix} - \left(-\mu + \vec{\alpha}^T f\kappa\right)\vec{\alpha}^T f^T \\ + \left(A\kappa^T + \tau^T + \vec{\alpha}^T f\sigma\right)\Gamma^T \end{matrix} \\ f\vec{\alpha}\mu + \Gamma\tau + \Gamma\kappa A - \left(f\vec{\alpha}\kappa^T + \Gamma\sigma\right) f^T\vec{\alpha} & \Gamma\kappa & -\Gamma\kappa\vec{\alpha}^T f^T + \left(f\vec{\alpha}\kappa^T + \Gamma\sigma\right)\Gamma^T \end{pmatrix}, \tag{3.72}$$

here $f\tau$ is a column vector and $\vec{\alpha}^T f$ is a row vector.

Considering $\left(\vec{\alpha}^T f\tau\right)^T = \vec{\alpha}^T f\tau$, Taking $\left(\vec{\alpha}^T f\right).\left(\vec{\alpha}^T f\right)^T = \left(\vec{\alpha}^T f\right)^2$, also since $\left(\vec{\alpha}^T f\right)\sigma\left(\vec{\alpha}^T f\right)^T$ is anti symmetric, that is,

$$\left(\vec{\alpha}^T f\right) = \begin{pmatrix} a_1 & a_2 \end{pmatrix} \begin{pmatrix} 0 & b_1 \\ -b_1 & 0 \end{pmatrix} \begin{pmatrix} a_1 & a_2 \end{pmatrix} = -a_2 b_1 a_1 + a_1 b_1 a_2 = 0.$$

We find

$$D_x\left(A - \frac{1}{2}\left(\vec{\alpha}^T f\right)^2\right) = D_x\left(\frac{\left(\vec{\alpha}^T f\right).\left(\vec{\alpha}^T f\right)^T}{2} - \frac{\left(\vec{\alpha}^T f\right).\left(\vec{\alpha}^T f\right)^T}{2}\right) = 0.$$

we obtain

$$\begin{aligned} & D_x\left(f\vec{\alpha}\right) - D_x\Gamma\left(f^T\vec{\alpha}\right) - \left(D_x\left(\vec{\alpha}^T f\right)\Gamma^T\right)^T \\ & = D_x\left(f\vec{\alpha}\right) - D_x\left(\Gamma f^T\vec{\alpha}\right) \\ & = D_x\left(f\vec{\alpha} - \Gamma f^T\vec{\alpha}\right) \\ & = D_x\left(\left(f - \Gamma f^T\right)\vec{\alpha}\right), \\ & = D_x\left(\left(f - f^T - \underline{\phi} f f^T\right)\vec{\alpha}\right) = D_x(0) = 0. \end{aligned} \tag{3.73}$$

And

$$
\begin{aligned}
\tilde{\sigma}^T &= \Gamma D_x \Gamma^T - f\vec{\alpha}\kappa^T\Gamma^T + \Gamma\kappa\alpha^T f^T + \Gamma\sigma^T\Gamma^T \\
\tilde{\sigma} + \tilde{\sigma}^T &= D_x\Gamma\Gamma^T + \Gamma D_x\Gamma^T + 0 \\
&= D_x\left(\Gamma\Gamma^T\right) = D_x\left(\Gamma\Gamma^{-1}\right) = D_x\left(I\right) = 0.
\end{aligned} \tag{3.74}
$$

Then we obtain the Cartan matrix which has gauge freedom given by $x$-dependent $SO(n,1)$ rotations and boosts,

$$
\widetilde{K} = \begin{pmatrix} \mu - \kappa^T f^T \vec{\alpha} & 0 & \kappa^T\Gamma^T \\ 0 & -\mu + \vec{\alpha}^T f\kappa & \begin{array}{c} \left(D_x\left(\vec{\alpha}^T f\right)\Gamma^T\right)^T + \mu\vec{\alpha}^T f^T - \vec{\alpha}^T f\kappa\vec{\alpha}^T f^T \\ + \left(A\kappa^T + \tau^T + \vec{\alpha}^T f\sigma\right)\Gamma^T \end{array} \\ \begin{array}{c} D_x\left(f\vec{\alpha}\right) - D_x\Gamma\left(f^T\vec{\alpha}\right) + f\vec{\alpha}\mu - f\vec{\alpha}\kappa^T f^T\vec{\alpha} \\ +A\Gamma\kappa + \Gamma\tau - \Gamma\sigma f^T\vec{\alpha} \end{array} & \Gamma\kappa & D_x\Gamma\Gamma^T - \Gamma\kappa\vec{\alpha}^T f^T + \left(f\vec{\alpha}\kappa^T + \Gamma\sigma\right)\Gamma^T \end{pmatrix}. \tag{3.75}
$$

We can write

$$
\begin{pmatrix} e_+ \\ \tilde{e}_- \\ \tilde{e}_{i\perp} \end{pmatrix}_x = \widetilde{K} \begin{pmatrix} e_+ \\ \tilde{e}_- \\ \tilde{e}_{i\perp} \end{pmatrix}, \tag{3.76}
$$

rearranging (3.76) such that

$$
\begin{pmatrix} e_+ \\ \tilde{e}_- \\ \tilde{e}_{i\perp} \end{pmatrix}_x = \begin{pmatrix} \widetilde{\mu} & 0 & \widetilde{\kappa}^T \\ 0 & -\widetilde{\mu} & \widetilde{\tau}^T \\ \widetilde{\tau} & \widetilde{\kappa} & \widetilde{\sigma} \end{pmatrix} \begin{pmatrix} e_+ \\ \tilde{e}_- \\ \tilde{e}_{i\perp} \end{pmatrix}, \quad \widetilde{\sigma}^T = -\widetilde{\sigma}, \tag{3.77}
$$

where $\widetilde{\mu}$ is scalar, $\widetilde{\kappa}$ and $\widetilde{\tau}$ are vectors and $\widetilde{\sigma}$ is antisymmetric matrix.

**Lemma 3.2.1.** The Frenet frame of null curve in terms of re-parameterization is given by equation (3.77), where $\widetilde{K}$ satisfies equation (3.75).

Now we show that our gauge transformation, what we used, is true. Firstly, we find $\tilde{e}_{0x}$, $\tilde{e}_{1x}$ and $\tilde{e}_{1x}$ by considering

$$
\mu\left(\tilde{e}_+, \tilde{e}_-\right) = -1, \mu\left(\tilde{e}_\pm, \tilde{e}_\pm\right) = 0 \tag{3.78}
$$

$$\tilde{e}_0 = \lambda\left(\tilde{e}_+ + \tilde{e}_-\right), \text{ timelike}$$

$$\mu\left(\tilde{e}_0, \tilde{e}_0\right) = \lambda^2\left(-2\right) \Rightarrow -2\lambda^2 = -1 \Rightarrow \lambda = \frac{1}{\sqrt{2}}.$$

$$\tilde{e}_1 = \lambda\left(\tilde{e}_+ - \tilde{e}_-\right), \text{ spacelike}$$

$$\mu\left(\tilde{e}_1, \tilde{e}_1\right) = -\lambda^2\left(-2\right) \Rightarrow 2\lambda^2 = 1 \Rightarrow \lambda = \frac{1}{\sqrt{2}}.$$

Then, we can write

$$\begin{pmatrix} \tilde{e}_0 \\ \tilde{e}_1 \\ \tilde{e}_{i\perp} \end{pmatrix}_x = \underline{K} \begin{pmatrix} \tilde{e}_0 \\ \tilde{e}_1 \\ \tilde{e}_{i\perp} \end{pmatrix}$$

To calculate $\underline{K}$ , we find $\tilde{e}_{0x}$, $\tilde{e}_{1x}$ and $\tilde{e}_{i\perp x}$.

To get $\tilde{e}_{0x}$, we take the $x$-derivative of $\tilde{e}_0 = \lambda\left(\tilde{e}_+ + \tilde{e}_-\right)$by using (3.78) which yields

$$\begin{aligned}
\tilde{e}_{0x} &= \lambda\left(\tilde{e}_{+x} + \tilde{e}_{-x}\right) = \lambda\left(\tilde{\mu}\tilde{e}_+ + \tilde{\kappa}^T\tilde{e}_\perp - \tilde{\mu}\tilde{e}_- + \tilde{\tau}^T\tilde{e}_\perp\right) \\
&= \lambda\left(\tilde{\mu}\left(\tilde{e}_+ - \tilde{e}_-\right) + \left(\tilde{\kappa}^T + \tilde{\tau}^T\right)\tilde{e}_\perp\right) \\
&= \tilde{\mu}\tilde{e}_1 + \lambda\left(\tilde{\kappa} + \tilde{\tau}\right)^T\tilde{e}_\perp.
\end{aligned}$$

To obtain $\tilde{e}_{1x}$, we differentiate $\tilde{e}_1 = \lambda\left(\tilde{e}_+ - \tilde{e}_-\right)$ in terms of the $x$ by using (3.78) which yields

$$\begin{aligned}
\tilde{e}_{1x} &= \lambda\left(\tilde{e}_{+x} - \tilde{e}_{-x}\right) = \lambda\left(\tilde{\mu}\tilde{e}_+ + \tilde{\kappa}^T\tilde{e}_\perp - \left(-\tilde{\mu}\tilde{e}_- + \tilde{\tau}^T\tilde{e}_\perp\right)\right) \\
&= \lambda\left(\tilde{\mu}\left(\tilde{e}_+ + \tilde{e}_-\right) + \left(\tilde{\kappa}^T - \tilde{\tau}^T\right)\tilde{e}_\perp\right) \\
&= \tilde{\mu}\tilde{e}_0 + \lambda\left(\tilde{\kappa} - \tilde{\tau}\right)^T\tilde{e}_\perp.
\end{aligned}$$

$$\tilde{e}_+ = \frac{1}{2\lambda}\left(\tilde{e}_0 + \tilde{e}_1\right), \ \tilde{e}_- = \frac{1}{2\lambda}\left(\tilde{e}_0 - \tilde{e}_1\right). \tag{3.79}$$

To obtain $\left(\tilde{e}_{i\perp}\right)_x$ by using (3.78) which yields

$$\begin{aligned}
\left(\tilde{e}_{i\perp}\right)_x &= \frac{\tilde{\tau}}{2\lambda}\left(\tilde{e}_0 + \tilde{e}_1\right) + \frac{\tilde{\kappa}}{2\lambda}\left(\tilde{e}_0 - \tilde{e}_1\right) + \tilde{\sigma}\tilde{e}_\perp \\
&= \frac{1}{2\lambda}\left(\tilde{\kappa} + \tilde{\tau}\right)\tilde{e}_0 + \frac{1}{2\lambda}\left(\tilde{\tau} - \tilde{\kappa}\right)\tilde{e}_1 + \tilde{\sigma}\tilde{e}_\perp.
\end{aligned}$$

Then we can write

$$\begin{pmatrix} \tilde{e}_0 \\ \tilde{e}_1 \\ \tilde{e}_{i\perp} \end{pmatrix}_x = \begin{pmatrix} 0 & \tilde{\mu} & \lambda\left(\tilde{\kappa}+\tilde{\tau}\right) \\ \tilde{\mu} & 0 & \lambda\left(\tilde{\kappa}-\tilde{\tau}\right)^T \\ \frac{1}{2\lambda}\left(\tilde{\kappa}+\tilde{\tau}\right) & \frac{1}{2\lambda}\left(\tilde{\tau}-\tilde{\kappa}\right) & \tilde{\sigma} \end{pmatrix} \begin{pmatrix} \tilde{e}_0 \\ \tilde{e}_1 \\ \tilde{e}_\perp \end{pmatrix}, \tag{3.80}$$

we can calculate

$$\begin{aligned} tr\underline{K}^2 &= tr\left(\begin{pmatrix} 0 & \tilde{\mu} & \lambda\left(\tilde{\kappa}+\tilde{\tau}\right)^T \\ \tilde{\mu} & 0 & \lambda\left(\tilde{\kappa}-\tilde{\tau}\right)^T \\ \frac{1}{2\lambda}\left(\tilde{\kappa}+\tilde{\tau}\right) & \frac{1}{2\lambda}\left(\tilde{\tau}-\tilde{\kappa}\right) & \tilde{\sigma} \end{pmatrix}\begin{pmatrix} 0 & \tilde{\mu} & \lambda\left(\tilde{\kappa}+\tilde{\tau}\right)^T \\ \tilde{\mu} & 0 & \lambda\left(\tilde{\kappa}-\tilde{\tau}\right)^T \\ \frac{1}{2\lambda}\left(\tilde{\kappa}+\tilde{\tau}\right) & \frac{1}{2\lambda}\left(\tilde{\tau}-\tilde{\kappa}\right) & \tilde{\sigma} \end{pmatrix}\right) \\ &= tr\begin{pmatrix} \tilde{\mu}^2+\frac{1}{2}\left(\tilde{\kappa}+\tilde{\tau}\right)^T\left(\tilde{\kappa}+\tilde{\tau}\right) & ... & ... \\ ... & \tilde{\mu}^2+\frac{1}{2}\left(\tilde{\kappa}-\tilde{\tau}\right)^T\left(\tilde{\tau}-\tilde{\kappa}\right) & ... \\ ... & ... & \begin{array}{c}\frac{1}{2}\left(\tilde{\kappa}+\tilde{\tau}\right)\left(\tilde{\kappa}+\tilde{\tau}\right)^T \\ +\frac{1}{2}\left(\tilde{\tau}-\tilde{\kappa}\right)\left(\tilde{\kappa}-\tilde{\tau}\right)^T+\tilde{\sigma}\tilde{\sigma}\end{array} \end{pmatrix} \end{aligned}$$

$$\begin{aligned} &= 2\tilde{\mu}^2+\frac{1}{2}\left[\left(\tilde{\kappa}+\tilde{\tau}\right)^T\left(\tilde{\kappa}+\tilde{\tau}\right)+\left(\tilde{\kappa}-\tilde{\tau}\right)^T\left(\tilde{\tau}-\tilde{\kappa}\right)\right] \\ &+ tr\left[\frac{1}{2}\left(\left(\tilde{\kappa}+\tilde{\tau}\right)\left(\tilde{\kappa}+\tilde{\tau}\right)^T+\left(\tilde{\tau}-\tilde{\kappa}\right)\left(\tilde{\kappa}-\tilde{\tau}\right)^T\right)+\tilde{\sigma}\tilde{\sigma}\right] \\ &= 2\tilde{\mu}^2+\left(\tilde{\kappa}+\tilde{\tau}\right)^T\left(\tilde{\kappa}+\tilde{\tau}\right)+tr\left(\tilde{\sigma}\tilde{\sigma}\right). \end{aligned}$$

taking $tr\left(\kappa\tau^T\right)=\kappa^T\tau$

$$\begin{aligned} tr\left(\underline{K}\underline{K}^T\right) &= tr\left(\begin{pmatrix} 0 & \tilde{\mu} & \lambda\left(\tilde{\kappa}+\tilde{\tau}\right)^T \\ \tilde{\mu} & 0 & \lambda\left(\tilde{\kappa}-\tilde{\tau}\right)^T \\ \frac{1}{2\lambda}\left(\tilde{\kappa}+\tilde{\tau}\right) & \frac{1}{2\lambda}\left(\tilde{\tau}-\tilde{\kappa}\right) & \tilde{\sigma} \end{pmatrix}\begin{pmatrix} 0 & \tilde{\mu} & \frac{1}{2\lambda}\left(\tilde{\kappa}+\tilde{\tau}\right)^T \\ \tilde{\mu} & 0 & \frac{1}{2\lambda}\left(\tilde{\tau}-\tilde{\kappa}\right)^T \\ \lambda\left(\tilde{\kappa}+\tilde{\tau}\right) & \lambda\left(\tilde{\kappa}-\tilde{\tau}\right) & \tilde{\sigma}^T \end{pmatrix}\right) \\ &= tr\begin{pmatrix} \tilde{\mu}^2+\lambda^2\left(\tilde{\kappa}+\tilde{\tau}\right)^T\left(\tilde{\kappa}+\tilde{\tau}\right) & ... & ... \\ ... & \tilde{\mu}^2+\lambda^2\left(\tilde{\kappa}-\tilde{\tau}\right)^T\left(\tilde{\kappa}-\tilde{\tau}\right) & ... \\ ... & ... & \begin{array}{c}\frac{1}{4\lambda^2}\left(\tilde{\kappa}+\tilde{\tau}\right)\left(\tilde{\kappa}+\tilde{\tau}\right)^T \\ +\frac{1}{4\lambda^2}\left(\tilde{\tau}-\tilde{\kappa}\right)\left(\tilde{\tau}-\tilde{\kappa}\right)^T+\tilde{\sigma}\tilde{\sigma}^T\end{array} \end{pmatrix} \end{aligned}$$

$$
\begin{aligned}
&= 2\tilde{\mu}^2 + \lambda^2 \left[ (\tilde{\kappa}+\tilde{\tau})^T (\tilde{\kappa}+\tilde{\tau}) + (\tilde{\kappa}-\tilde{\tau})^T (\tilde{\kappa}-\tilde{\tau}) \right] \\
&\quad + tr \left[ \frac{1}{4\lambda^2} \left( (\tilde{\kappa}+\tilde{\tau}) (\tilde{\kappa}+\tilde{\tau})^T + (\tilde{\tau}-\tilde{\kappa}) (\tilde{\tau}-\tilde{\kappa})^T \right) + \tilde{\sigma}\tilde{\sigma} \right] \\
&= 2\tilde{\mu}^2 + \left( \lambda^2 + \frac{1}{4\lambda^2} \right) (\tilde{\kappa}+\tilde{\tau})^T (\tilde{\kappa}+\tilde{\tau}) + \left( \lambda^2 - \frac{1}{4\lambda^2} \right) (\tilde{\kappa}-\tilde{\tau})^T (\tilde{\kappa}-\tilde{\tau}) + tr\,(\tilde{\sigma}\tilde{\sigma})\,.
\end{aligned}
$$

### 3.2.2 Gauge Choices

We will show whether our choose is true or not. To show that we use three ways as follows,

1)

$$
\begin{aligned}
\tilde{\sigma} = 0 &= D_x \Gamma \Gamma^T - \Gamma \kappa \vec{\alpha}^T f^T + \left( \Gamma \kappa \vec{\alpha}^T f^T \right)^T + \Gamma \sigma^T \Gamma^T \\
&= D_x \Gamma - \Gamma \kappa \vec{\alpha}^T f^T + \Gamma \kappa \vec{\alpha}^T f^T \Gamma + \Gamma \sigma, \qquad (3.81)
\end{aligned}
$$

where $\Gamma = 1 + \underline{\phi} f$, $\Gamma^T = -\Gamma$, so we can solve this equations.

2)

$$
\tilde{\kappa} = 0 = \kappa \Gamma \Rightarrow \kappa = 0, \qquad (3.82)
$$

so it is a contradiction, thus we can not use this choose.

2′)

$$
\tilde{\tau} = 0 = f \vec{\alpha}_x + \left( D_x f - D_x \Gamma f^T - \Gamma \sigma f^T + \mu f \right) \vec{\alpha} + \frac{1}{2} \left( \vec{\alpha}^T f f^T \vec{\alpha} \right) \Gamma \kappa - \left( \vec{\alpha}^T f \kappa \right) f \vec{\alpha} + \Gamma \tau, \qquad (3.83)
$$

here $f^{-1}$ is given by

$$
0 = \vec{\alpha}_x + f^{-1} \left( D_x f - D_x \Gamma f^T - \Gamma \sigma f^T + \mu f \right) \vec{\alpha} + \frac{1}{2} \left( \vec{\alpha}^T f f^T \vec{\alpha} \right) f^{-1} \Gamma \kappa - \left( \vec{\alpha}^T f \kappa \right) \vec{\alpha} + f^{-1} \Gamma \tau. \qquad (3.84)
$$

We can solve this equation. Then our gauge choose is true.

## 4 GEOMETRIC FLOWS of NULL CURVE by USING ALGEBRAIC WAY in LORENTZIAN SPACE

In this section, we find the flow for null curves by using different ways. In subsection 4.1, we consider scaling and higher symmetry to find the Burger's flows. The rest of this section, we use the gauge transformation to find the Gradient, the Airy and the Hamiltonian flows.

### 4.1 Burger's Flows

If we consider $h_{\|} = u$ and $\overrightarrow{h_{\perp}} = 2\overrightarrow{u}$ or $\overrightarrow{h_{\perp}} = -2\overrightarrow{u}$ in (3.34)-(3.38) we will find two kinds of Burger's equations by using scaling $x \to \lambda x$, $u \to \lambda^{-1} u$ and $\overrightarrow{u} \to \lambda^{-1}\overrightarrow{u}$, higher symmetry as follows

$i)$ For $\overrightarrow{h_{\perp}} = 2\overrightarrow{u}$,

$$u_t = u_{xx} + 2uu_x, \tag{4.1}$$

$$\overrightarrow{u}_t = -\overrightarrow{u}_{xx} + 2u\overrightarrow{u}_x + 2u_x\overrightarrow{u}, \tag{4.2}$$

$ii)$ For $\overrightarrow{h_{\perp}} = -2\overrightarrow{u}$,

$$u_t = u_{xx} + 2uu_x, \tag{4.3}$$

$$\overrightarrow{u}_t = \overrightarrow{u}_{xx} + 2u_x\overrightarrow{u}, \tag{4.4}$$

where Burger's Recursion operator

$$R = D_x + u + u_x D_x^{-1}.$$

If we make the gauge transformation as follows

$$\begin{aligned} u = U_x\,, w = U_t \qquad v = e^U \\ \overrightarrow{u} = e^U \vec{U}_x\,, \qquad \overrightarrow{v} = e^U \vec{U} \end{aligned} \tag{4.5}$$

and using (3.38) we obtain

$$D_t\left(e^{-U}\overrightarrow{u}\right) = e^{-U}\left(\overrightarrow{u}_t - U_t\overrightarrow{u}\right).$$

By taking into the above equation account, we write

$$D_x\left(e^{-U}\overrightarrow{w}\right) = e^{-U}\left(\vec{w}_x - U_x\vec{w}\right).$$

Then, we can determine

$$e^{-U}\overrightarrow{u} = \vec{U}_x,\ e^{-U}\overrightarrow{w} = \vec{U}_t.$$

Now, we can rearrange (3.34) and (3.35) equations in terms of $u, \overrightarrow{u}$ and $U, \vec{U}$ as follows

$$w = D_x h_{\|} + U_x h_{\|} = e^{-U}\left(e^U h_{\|}\right),$$

$$\vec{w} = -\frac{1}{2}D_x\overrightarrow{h_\perp} + h_{\|}e^U\overrightarrow{U_x},$$

then (3.37) and (3.38) seem like

$$U_t = D_x h_{\|} + U_x h_{\|} = e^{-U}D_x\left(e^U h_{\|}\right) \tag{4.6}$$

$$e^U\vec{U}_t = -\frac{1}{2}D_x\overrightarrow{h_\perp} + h_{\|}e^U\overrightarrow{U_x}. \tag{4.7}$$

Hence, we get the following equation using (4.6)

$$\left(e^U\right)_t = D_x\left(e^U h_{\|}\right) \tag{4.8}$$

Also, we write

$$\left(e^U\vec{U}\right)_t = e^U\vec{U}_t + \vec{U}e^U U_t.$$

If we substitute (4.6) and (4.7) into the above equation, then we get

$$\left(e^U\vec{U}\right)_t = D_x\left(-\frac{1}{2}\overrightarrow{h_\perp}\right) + \left(e^U h_{\|}\vec{U}\right)_x, \tag{4.9}$$

$$\left(e^U\vec{U}\right)_t = D_x\left(-\frac{1}{2}\overrightarrow{h_\perp} + \left(e^U h_{\|}\vec{U}\right)_x\right). \tag{4.10}$$

Lastly, $u$ and $\overrightarrow{u}$ vectors are given in terms of $v$, $\overrightarrow{v}$ and $U, \vec{U}$ as follows:

$$\begin{array}{ll} U = \ln v\,, & \vec{U} = e^{-U}\overrightarrow{v} = \dfrac{\overrightarrow{v}}{v} \\ u = (\ln v)_x = \dfrac{v_x}{v} & \overrightarrow{u} = v\left(\dfrac{\overrightarrow{v}}{v}\right)_x = \overrightarrow{v}_x - \dfrac{v_x}{v}\overrightarrow{v} \end{array}\,. \tag{4.11}$$

## 4.2 Gradient flows

In this subsection, we present the Gradient flows for the components $v$, $\vec{v}$ which obtained the Gauge transformation (4.11) and the natural components $u$, $\vec{u}$, respectively.

### $i)$ Gradient flows of the components $v$ and $\vec{v}$

Considering

$$v_t = -\frac{\delta H}{\delta v},\ \vec{v}_t = -\frac{\delta H}{\delta \vec{v}}, \tag{4.12}$$

where

$$H = \frac{1}{2}v_x^2 + \frac{1}{2}\vec{v}_x^2. \tag{4.13}$$

Also, we can determine

$$\tilde{h}_{\|} = v h_{\|}, \overrightarrow{\tilde{h}_{\perp}} = -\frac{1}{2}\overrightarrow{h_{\perp}} + \frac{\tilde{h}_{\|}}{v}\vec{v}, \tag{4.14}$$

where $h_{\|} = u$ and $\overrightarrow{h_{\perp}} = -2\overrightarrow{u}$.

Then, substituting (4.14) into (4.8) and (4.10) , we obtain the evolution equations as follows

$$\begin{pmatrix} v \\ \vec{v} \end{pmatrix}_t = D_x \begin{pmatrix} \tilde{h}_{\|} \\ \overrightarrow{\tilde{h}_{\perp}} \end{pmatrix} = \begin{pmatrix} v_{xx} \\ \vec{v}_{xx} \end{pmatrix}, \tag{4.15}$$

where $D_x$ is the Recursion operator.

Now, we will show these equations are the gradient equations in terms of $v$, $\vec{v}$ components. To this end, we can write

$$D_t \int H dx = \int D_t H dx \tag{4.16}$$

and

$$
\begin{aligned}
D_t H &= H_v v_t + H_{v_x} v_{t_x} + ... \\
&+ H_{\vec{v}} . \vec{v}_t + H_{\vec{v}x} . \vec{v}_{tx} + ... \\
&= v_t \frac{\delta H}{\delta v} + \vec{v}_t \frac{\delta H}{\delta \vec{v}} + D_x(...).
\end{aligned}
$$

If we substitute the above equation into (4.16), we obtain

$$
D_t \int H dx = \int \left( v_t \frac{\delta H}{\delta v} + \vec{v}_t \frac{\delta H}{\delta \vec{v}} \right) dx \qquad \text{mod boundary terms.} \tag{4.17}
$$

Substituting (4.12) into (4.17), we obtain

$$
\begin{aligned}
D_t \int H dx &= - \int \left( (\frac{\delta H}{\delta v})^2 + (\frac{\delta H}{\delta \vec{v}})^2 \right) dx < 0 \\
&\Longrightarrow \frac{d}{dt} \int H dx < 0.
\end{aligned}
$$

If $H > 0$, then $\int H dx < 0 \Longrightarrow H \to 0$. Thus we find the gradient flow.

$ii)$ **Gradient flows of the components $u$ and $\vec{u}$**

We find the Gradient flows in terms of natural components $u, \vec{u}$. To this end, we can write

$$
\delta H = \frac{\delta H}{\delta u} \delta u + \frac{\delta H}{\delta \overrightarrow{u}} \delta \overrightarrow{u}, \tag{4.18}
$$

where $\delta u$ and $\delta \overrightarrow{u}$ are given by the following equations

$$
\begin{aligned}
\delta u = \frac{\delta u}{\delta v} \delta v &= \left( \frac{\partial u}{\partial v} - D_x \frac{\partial u}{\partial v_x} \right) \delta v, \qquad (4.19) \\
&= \left( -\frac{v_x}{v^2} - D_x \left( \frac{1}{v} \right) \right) \delta v.
\end{aligned}
$$

Taking

$$
D_x \left( \frac{1}{v} \right) \delta v = -\frac{1}{v} D_x \delta v,
$$

we obtain

$$\delta u = \frac{1}{v} D_x \delta v - \frac{v_x}{v^2} \delta v = Dx(\frac{1}{v}\delta v), \tag{4.20}$$

$$\begin{aligned}
\delta \overrightarrow{u} &= \frac{\delta \overrightarrow{u}}{\delta v}\delta v + \frac{\delta \overrightarrow{u}}{\delta \vec{v}}\delta \vec{v} = \left(-\frac{v_x}{v^2}\vec{v} - D_x\left(\frac{1}{v}\vec{v}\right)\right)\delta v - \frac{v_x}{v}\delta \vec{v} + D_x \vec{v}, \\
\delta \overrightarrow{u} &= D_x \delta \vec{v} - \frac{1}{v} D_x \delta v \vec{v} + \frac{v_x}{v^2}\delta v \vec{v} - \frac{v_x}{v}\delta \vec{v}, \\
&= D_x \delta \vec{v} - D_x(\frac{1}{v}\delta v)\vec{v} - \frac{v_x}{v}\delta \vec{v}.
\end{aligned} \tag{4.21}$$

If we substitute (4.20) and (4.21) into (4.18) to obtain

$$\begin{aligned}
\delta H &= \frac{\delta H}{\delta u}(\frac{1}{v}D_x \delta v - \frac{v_x}{v^2}\delta v) + \frac{\delta H}{\delta \overrightarrow{u}}(D_x \delta \vec{v} - D_x(\frac{1}{v}\delta v)\vec{v} - \frac{v_x}{v}\delta \vec{v}), \\
&= \delta v(-\frac{v_x}{v^2}\frac{\delta H}{\delta u} - D_x(\frac{1}{v}\frac{\delta H}{\delta u}) + \frac{v_x}{v^2}\vec{v}.\frac{\delta H}{\delta \overrightarrow{u}} + D_x(\frac{1}{v}\vec{v}.\frac{\delta H}{\delta \overrightarrow{u}})) \\
&+ \delta \vec{v}(-D_x \frac{\delta H}{\delta \overrightarrow{u}} - \frac{v_x}{v}\frac{\delta H}{\delta \overrightarrow{u}}), \\
\delta H &= \delta v(-\frac{1}{v}D_x \frac{\delta H}{\delta u} + \frac{1}{v}D_x(\vec{v}.\frac{\delta H}{\delta \overrightarrow{u}})) + \delta \vec{v}(-D_x \frac{\delta H}{\delta \overrightarrow{u}} - \frac{v_x}{v}\frac{\delta H}{\delta \overrightarrow{u}}).
\end{aligned} \tag{4.22}$$

Then, from (4.22), we get

$$\frac{\delta H}{\delta v} = -\frac{1}{v}D_x \frac{\delta H}{\delta u} + \frac{1}{v}D_x(\vec{v}.\frac{\delta H}{\delta \overrightarrow{u}}), \tag{4.23}$$

$$\frac{\delta H}{\delta \vec{v}} = -D_x \frac{\delta H}{\delta \overrightarrow{u}} - \frac{v_x}{v}\frac{\delta H}{\delta \overrightarrow{u}}. \tag{4.24}$$

From (4.11) and (4.12), $u_t$ is defined by

$$\begin{aligned}
u_t &= \left(\frac{v_x}{v}\right)_t = \frac{v_{xt}}{v} - \frac{v_x v_t}{v^2} = \left(\frac{v_t}{v}\right)_x \\
&= D_x\left(\frac{1}{v}\left(-\frac{\delta H}{\delta v}\right)\right).
\end{aligned} \tag{4.25}$$

Substituting (4.23) into (4.25), we get

$$u_t = D_x\left(\frac{1}{v}\left(\frac{1}{v}D_x \frac{\delta H}{\delta u} - \frac{1}{v}D_x(\vec{v}.\frac{\delta H}{\delta \overrightarrow{u}})\right)\right), \tag{4.26}$$

and by using (4.11) and (4.12), $\overrightarrow{u}_t$ is defined by

$$\begin{aligned}\overrightarrow{u}_t &= \vec{v}_{tx-}\left(\frac{v_x}{v}\right)\vec{v}_{t-}\left(\frac{v_x}{v}\right)_t\vec{v},\\ &= D_x\left(-\frac{\delta H}{\delta\vec{v}}\right)-\frac{v_x}{v}\left(-\frac{\delta H}{\delta\vec{v}}\right)-u_t\vec{v},\end{aligned}\tag{4.27}$$

substituting (4.24) into (4.27), we obtain

$$\begin{aligned}\overrightarrow{u}_t &= D_x^2\frac{\delta H}{\delta\overrightarrow{u}}+D_x\left(\frac{v_x}{v}\frac{\delta H}{\delta\overrightarrow{u}}\right)+\frac{v_x}{v}\left(-D_x\frac{\delta H}{\delta\overrightarrow{u}}-\frac{v_x}{v}\frac{\delta H}{\delta\overrightarrow{u}}\right)-u_t\vec{v},\\ &= D_x^2\frac{\delta H}{\delta\overrightarrow{u}}+\left(\left(\frac{v_x}{v}\right)_x-\left(\frac{v_x}{v}\right)^2\right)\frac{\delta H}{\delta\overrightarrow{u}}-u_t\vec{v},\\ &= D_x^2\frac{\delta H}{\delta\overrightarrow{u}}+\left(\frac{v_{xv}}{v}-2\left(\frac{v_x}{v}\right)^2\right)\frac{\delta H}{\delta\overrightarrow{u}}-u_t\vec{v},\\ &= D_x^2\frac{\delta H}{\delta\overrightarrow{u}}+v\left(\frac{v_x}{v}\right)_x^2\frac{\delta H}{\delta\overrightarrow{u}}-u_t\vec{v},\\ &= D_x^2\frac{\delta H}{\delta\overrightarrow{u}}-v\left(\frac{1}{v}\right)_{xx}^2\frac{\delta H}{\delta\overrightarrow{u}}-D_x\left(\frac{1}{v}\left(\frac{1}{v}D_x\frac{\delta H}{\delta u}-\frac{1}{v}D_x(\vec{v}.\frac{\delta H}{\delta\overrightarrow{u}})\right)\right)\vec{v}.\end{aligned}\tag{4.28}$$

In the light of these above equations, $u_t, \vec{u}_t$ can be determined

$$\begin{pmatrix} u \\ \vec{u}\end{pmatrix}_t = -\mathcal{D}^*\mathcal{D}\begin{pmatrix}\frac{\delta H}{\delta u}\\ \frac{\delta H}{\delta\vec{u}}\end{pmatrix},\tag{4.29}$$

where

$$\mathcal{D}=\begin{pmatrix} D_1 & D_2\\ D_3 & D_4\end{pmatrix},$$

$$\mathcal{D}^*=\begin{pmatrix} D_1^* & D_2^*\\ D_3^* & D_4^*\end{pmatrix}^{\intercal}=\begin{pmatrix} D_1^* & D_3^*\\ D_2^* & D_4^*\end{pmatrix},$$

$$\mathcal{D}^*\mathcal{D}=\begin{pmatrix} D_1^*D_1+D_3^*D_3 & D_1^*D_2+D_3^*D_4\\ D_2^*D_1+D_4^*D_3 & D_2^*D_2+D_4^*D_4\end{pmatrix},$$

and $D_1$ is a scalar, $D_2$ and $D_3$ are vectors and $D_4$ is a matrix or scalar. If $\mathcal{D}$ is calculated, then we find

$$\mathcal{D} = \begin{pmatrix} \frac{1}{v}D_x & -\frac{1}{v}D_x\vec{v} \\ 0 & \frac{1}{v}D_x v \end{pmatrix}. \tag{4.30}$$

From the Gradient flow condition

$$\frac{d}{dt}\int H dx = \int \left( u_t \frac{\delta H}{\delta u} + \vec{u}_t \frac{\delta H}{\delta \vec{u}} \right) dx \ \text{ mod. boundary terms,} \tag{4.31}$$

if we consider

$$u_t = -\mathcal{D}^*\mathcal{D}\frac{\delta H}{\delta u} \text{ and } \vec{u}_t = -\mathcal{D}^*\mathcal{D}\frac{\delta H}{\delta \vec{u}}$$

in (4.31) equation, then we get

$$\begin{aligned} \frac{d}{dt}\int H dx &= -\int \left( \left( \mathcal{D}^*\mathcal{D}\frac{\delta H}{\delta u} \right) \frac{\delta H}{\delta u} + \left( \mathcal{D}^*\mathcal{D}\frac{\delta H}{\delta \vec{u}} \right) \frac{\delta H}{\delta \vec{u}} \right) dx, \\ &= -\int \left( \left( \mathcal{D}\frac{\delta H}{\delta u} \right) \left( \mathcal{D}\frac{\delta H}{\delta u} \right) + \left( \mathcal{D}\frac{\delta H}{\delta \vec{u}} \right) \left( \mathcal{D}\frac{\delta H}{\delta \vec{u}} \right) \right) dx, \\ \frac{d}{dt}\int H dx &< 0, \end{aligned} \tag{4.32}$$

so these equations become the gradient equations.

## 4.3 Airy Flows

If we consider

$$v_t = -D_x \frac{\delta H}{\delta v} = v_{3x}, \vec{v}_t = -D_x \frac{\delta H}{\delta \vec{v}} = \vec{v}_{3x}, \tag{4.33}$$

where

$$H = \frac{1}{2}v_x^2 + \frac{1}{2}\vec{v}_x^2,$$

then (4.33) yield

$$\begin{pmatrix} v \\ \vec{v} \end{pmatrix}_t = D_x \begin{pmatrix} -\frac{\delta H}{\delta v} \\ -\frac{\delta H}{\delta \vec{v}} \end{pmatrix}, \tag{4.34}$$

which are the Airy equations. Here

$$\begin{pmatrix} D_x & 0 \\ 0 & D_x \end{pmatrix} = D_x, \quad D^* = -D.$$

## 4.4 Hamiltonian Flows

We consider

$$\begin{pmatrix} u \\ \vec{u} \end{pmatrix}_t = \mathcal{H} \begin{pmatrix} \frac{\delta H}{\delta u} \\ \frac{\delta H}{\delta \vec{u}} \end{pmatrix}, \tag{4.35}$$

where

$$\mathcal{H} = \begin{pmatrix} \mathcal{H}_1 & \mathcal{H}_2 \\ \mathcal{H}_3 & \mathcal{H}_4 \end{pmatrix},$$

and $\mathcal{H}_1$ is a scalar, $\mathcal{H}_2$ and $\mathcal{H}_3$ are vectors and $\mathcal{H}_4$ is a matrix or scalar. From (4.33), $\mathcal{H}$ is given by

$$\mathcal{H} = \begin{pmatrix} D_x \frac{1}{v} D_x \frac{1}{v} D_x & -D_x \frac{1}{v} D_x \frac{1}{v} D_x (\vec{v}.) \\ -\vec{v} D_x \frac{1}{v} D_x \frac{1}{v} D_x & \begin{matrix} D_x^3 + \left(2\frac{v_{xx}}{v} - 3\frac{v_x^2}{v^2}\right) D_x + \\ \left(\left(\frac{v_x}{v}\right)_{xx} - \frac{v_x}{v}\left(\frac{v_x}{v}\right)_x + \vec{v} D_x \frac{1}{v} D_x \frac{1}{v} D_x (\vec{v}.)\right) \end{matrix} \end{pmatrix}, \quad \mathcal{H}^* = -\mathcal{H} \tag{4.36}$$

where we can easily show that $\mathcal{H}$ is the anti-self adjoint, hence we can say that $\mathcal{H}$ is the Hamiltonian operator.

To find $\mathcal{H}$, substituting (4.23), (4.24) and (4.33) into (4.11), respectively, we find

$$u_t = \left(\frac{v_x}{v}\right)_x = \frac{(v_t)_x}{v} - \frac{v_x v_t}{v^2} \tag{4.37}$$

$$= \frac{1}{v} D_x \left(-D_x \frac{\delta H}{\delta v}\right) - \frac{v_x}{v^2}\left(-D_x \frac{\delta H}{\delta v}\right) \tag{4.38}$$

$$= v D_x \left(-\frac{1}{v} D_x \left(\frac{\delta H}{\delta v}\right)\right) \tag{4.39}$$

$$= D_x \left(\frac{1}{v} D_x \left(\frac{1}{v} D_x \frac{\delta H}{\delta u} - \frac{1}{v} D_x \left(\vec{v}. \frac{\delta H}{\delta \overrightarrow{u}}\right)\right)\right) \tag{4.40}$$

$$= D_x \frac{1}{v} D_x \frac{1}{v} D_x \frac{\delta H}{\delta u} - D_x \frac{1}{v} D_x \frac{1}{v} D_x \left(\vec{v}. \frac{\delta H}{\delta \overrightarrow{u}}\right) \tag{4.41}$$

and

$$
\begin{aligned}
\overrightarrow{u}_t &= \vec{v}_{tx} - \left(\frac{v_x}{v}\right)_t \vec{v} - \frac{v_x}{v}\vec{v}_t \\
&= D_x\left(-D_x\frac{\delta H}{\delta\overrightarrow{v}}\right) - \frac{v_x}{v}\left(-D_x\frac{\delta H}{\delta\overrightarrow{v}}\right) - u_t\vec{v} \\
&= D_x^2\left(D_x\frac{\delta H}{\delta\overrightarrow{u}} + \frac{v_x}{v}\frac{\delta H}{\delta\overrightarrow{u}}\right) - \frac{v_x}{v}D_x\left(D_x\frac{\delta H}{\delta\overrightarrow{u}} + \frac{v_x}{v}\frac{\delta H}{\delta\overrightarrow{u}}\right) - u_t\vec{v} \\
&= D_x\left(D_x^2\frac{\delta H}{\delta\overrightarrow{u}} + \left(\frac{v_x}{v}\right)_x\frac{\delta H}{\delta\overrightarrow{u}} + \frac{v_x}{v}D_x\frac{\delta H}{\delta\overrightarrow{u}}\right) - \\
&\quad -\frac{v_x}{v}\left(D_x^2\frac{\delta H}{\delta\overrightarrow{u}} + \left(\frac{v_x}{v}\right)_x\frac{\delta H}{\delta\overrightarrow{u}} + \frac{v_x}{v}D_x\frac{\delta H}{\delta\overrightarrow{u}}\right) - u_t\vec{v} \\
&= D_x^3\frac{\delta H}{\delta\overrightarrow{u}} + \left(\frac{v_x}{v}\right)_{xx}\frac{\delta H}{\delta\overrightarrow{u}} + 2\left(\frac{v_x}{v}\right)_x D_x\frac{\delta H}{\delta\overrightarrow{u}} + \frac{v_x}{v}D_x^2\frac{\delta H}{\delta\overrightarrow{u}} \\
&\quad -\frac{v_x}{v}\left(D_x^2\frac{\delta H}{\delta\overrightarrow{u}} + \left(\frac{v_x}{v}\right)_x\frac{\delta H}{\delta\overrightarrow{u}} + \frac{v_x}{v}D_x\frac{\delta H}{\delta\overrightarrow{u}}\right) - u_t\vec{v} \\
&= D_x^3\frac{\delta H}{\delta\overrightarrow{u}} + \left(2\left(\frac{v_x}{v}\right)_x - \frac{v_x^2}{v^2}\right)D_x\frac{\delta H}{\delta\overrightarrow{u}} + \left(\left(\frac{v_x}{v}\right)_{xx} - \frac{v_x}{v}\left(\frac{v_x}{v}\right)_x\right)\frac{\delta H}{\delta\overrightarrow{u}} \\
&\quad + \left(D_x\frac{1}{v}D_x\frac{1}{v}D_x\left(\vec{v}.\frac{\delta H}{\delta\overrightarrow{u}}\right)\right)\vec{v} - \left(D_x\frac{1}{v}D_x\frac{1}{v}D_x\frac{\delta H}{\delta\overrightarrow{u}}\right)\vec{v}
\end{aligned}
$$

then we write

$$
\mathcal{H} = \begin{pmatrix} D_x\frac{1}{v}D_x\frac{1}{v}D_x & -D_x\frac{1}{v}D_x\frac{1}{v}D_x(\vec{v}.) \\ -\vec{v}D_x\frac{1}{v}D_x\frac{1}{v}D_x & \begin{array}{c} D_x^3 + \left(2\frac{v_{xx}}{v} - 3\frac{v_x^2}{v^2}\right)D_x + \\ \left(\left(\frac{v_x}{v}\right)_{xx} - \frac{v_x}{v}\left(\frac{v_x}{v}\right)_x + \vec{v}D_x\frac{1}{v}D_x\frac{1}{v}D_x(\vec{v}.)\right) \end{array} \end{pmatrix}.
$$

Also, we can easily show that $\mathcal{H}^* = -\mathcal{H}$.

## 5 CONCLUSIONS

The research was motivated by issues related to geometric curve flows of non-stretching null curve in $SO(n,1)/SO(n-1,1)$. There are several aims for future research related to the work presented in this dissertation. One of them is to show what kind of flows do we have in $SU(n,1)/SO(n,1)$ by using gauge transformation and considering scaling and higher symmetry. Another one is to improve our studies about geometric curve flows in Lorentzian space.

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