

On Special Submanifolds of an α – Kenmotsu Manifold

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Abstract

In this paper, we study some special type of submanifolds such as Z-umbilical submanifolds associated with an $\overline{M}$ – vector field Z on a submanifold M of an α – Kenmotsu manifold $\overline{M}$ and we find some interesting results.

Key words: Z-umbilical submanifolds, α – Kenmotsu manifolds

Bir α – Kenmotsu Manifoldunun Bazı Özel Alt Manifoldları Üzerine

Özet

Bu çalışmada bir α – Kenmotsu manifoldunun bir alt manifoldu üzerinde bir $\overline{M}$ – vektör alanı Z ile birleştirilmiş Z-umbilik alt manifold kavramlarını inceledik ve bir takım ilginç sonuçlara ulaştık.

Anahtar Sözcükler: Z-umbilik altmanifold, α – Kenmotsu manifold

1. Introduction

The concepts of Z-umbilical submanifolds and totally Z-geodesic submanifolds are defined by Nirmala S. Agashe and Mangala R. Chafle in 1991 [1]. Nirmala and Mangala generalize geodesic and umbilic which are well known in classical differential geometry and obtain characterizations of same. In their study, E. Kılıç, B. Şahin and R. Güneş study these concepts for a semi-Riemann hypersurface of a Lorentzian manifold [4].

In 1981, D. Janssens and L. Vanhecke define α – Kenmotsu manifolds [3]. These are trans-Sasakian of type $(0, \alpha)$ in J.A. Oubina's sense [5]. H. Öztürk, N. Aktan and C. Murathan study about this manifolds satisfying some curvature conditions [7].

We find that there isn't Z-umbilical submanifolds of an α – Kenmotsu manifold.

2. Material and Methods

Let $\overline{M}$ be a $(2n+1)$ -dimensional almost contact metric manifold together with a metric tensor g , a tensor field φ of type $(1,1)$, a vector field ξ and a 1-form η on M , satisfying:

$$\begin{aligned}\varphi^2 &= -I + \eta \otimes \xi \\ \varphi \xi &= 0 \\ \eta \circ \varphi &= 0 \\ \eta(\xi) &= 1 \\ \eta(X) &= g(X, \xi)\end{aligned}\tag{2.1}$$

$$\begin{aligned}g(\varphi X, \varphi Y) &= g(X, Y) - \eta(X)\eta(Y) \\ g(\varphi X, Y) &= -g(X, \varphi Y)\end{aligned}\tag{2.2}$$

for any vector fields X,Y on $\overline{M}$. If in addition, the following hold:

$$(\overline{\nabla}_X \varphi)Y = \alpha \{ -g(X, \varphi Y)\xi - \eta(Y)\varphi X \}\tag{2.3}$$

and

$$\overline{\nabla}_X \xi = \alpha(X - \eta(X)\xi)\tag{2.4}$$

α being a non-zero real constant, then the structure is called α -Kenmotsu structure [2]. On an α -Kenmotsu manifold $\overline{M}$, the following relation is held:

$$R(X, Y)\xi = \alpha^2[\eta(X)Y - \eta(Y)X] \quad (2.5)$$

Now, let M be a submanifold immersed in $\bar{M}$ ($\xi \in \chi(M)$). The Riemannian metric induced on M is denoted by the same symbol g . Let TM and $TM^\perp$ be the Lie algebra of vector fields tangential to M and normal to M respectively and ∇ be induced Levi-Civita connection on M , then the Gauss and Weingarten formulas are given by

$$\bar{\nabla}_X Y = \nabla_X Y + h(X, Y) \quad (2.6)$$

$$\bar{\nabla}_X V = -A_V X + \nabla_X^\perp V \quad (2.7)$$

for all vector fields X, Y in TM and V in $TM^\perp$, where $\nabla^\perp$ is the connection on the normal bundle $TM^\perp$, h is the second fundamental form and A_V is the Weingarten map associated with the vector field $V \in TM^\perp$ as

$$g(A_V X, Y) = g(h(X, Y), V) \quad [8] \quad (2.8)$$

Let $(\bar{M}, \varphi, \xi, \eta, g)$ be an α -Kenmotsu manifold and M be a submanifold immersed in $\bar{M}$. Suppose that Z is a $C^\infty \bar{M}$ -vector field defined on M . Decomposing Z uniquely into tangential and normal components, we have

$$Z = Z_T + Z^N \quad (2.9)$$

where Z_T is a tangent vector field on M and Z^N is a vector field of M which is normal to M . Covariant derivative of Z with respect to tangent vector field X on M , from (2.5) and (2.6) gives

$$\begin{aligned} \bar{\nabla}_X Z &= (\nabla_X Z - A_{Z^N} X) + (\nabla_X^\perp Z^N + h(X, Z_T)) \\ &= \tan \bar{\nabla}_X Z + \text{nor} \bar{\nabla}_X Z, \end{aligned} \quad (2.10)$$

where ∇ is the connection on M induced by the Riemannian connection $\bar{\nabla}$ of $\bar{M}$.

A vector field X on M has been called as a quasi-principal vector field of Z on M if $\tan \bar{\nabla}_X Z$ is codirectional with X [1]. A curve c in M whose tangent vector field in M is quasi-

principal vector field of Z on M , has been called as quasi-line of curvature of Z on M [1].

For a normal section V on M , if the second fundamental tensor A_V is everywhere proportional to the identity transformation I , i.e.,

$$A_V = \lambda I \quad (2.11)$$

for some real function λ on M , then V is called an umbilical section on M or M is said to be umbilical with respect to V [7].

If;

$$\tan \bar{\nabla}_X Z = \nabla_X Z - A_{Z^N} X = \lambda X \quad (2.12)$$

for every tangent vector field X on M , then a submanifold M of α -Kenmotsu manifold $\bar{M}$ is said to be Z -umbilical submanifold, where λ is a real function on M [1]. Hence a Z -umbilical submanifold M is characterized by the property that

- i- every tangent vector field on M is a quasi-principal vector field of Z on M ,
- ii- every curve on M is a quasi-line of curvature of Z with quasi-principal curvature at any point p on M equal to a constant [1].

A submanifold M is said to be totally Z -geodesic submanifold if $h_Z = 0$ identically on M .

2.1. Non-existence of Z -umbilical Submanifolds of an α -Kenmotsu Manifold

Theorem 2.1 Let $\bar{M}$ be an α -Kenmotsu manifold. In this case, there isn't any Z -umbilical submanifold of M ($\xi \in TM$).

Proof. Let $\bar{M}$ be an α -Kenmotsu manifold, M be Z -umbilical submanifold of M . That is;

$$\tan \bar{\nabla}_X Z = \nabla_X Z - A_{Z^N} X = \lambda X,$$

for all $X \in TM$ and $Z \in TM$, where λ is a differentiable function. From (2.4) and (2.5), if we consider ξ in TM , we find

$$\nabla_X \xi = \alpha(X - \eta(X)\xi) \quad (2.13)$$

and

$$h(X, \xi) = 0, \quad (2.14)$$

for all X in TM .

$$\begin{aligned} \bar{R}(X, Y)Z &= \bar{\nabla}_X \bar{\nabla}_Y Z - \bar{\nabla}_Y \bar{\nabla}_X Z - \bar{\nabla}_{[X, Y]} Z \\ &= \bar{\nabla}_X (\bar{\nabla}_Y Z_T + \bar{\nabla}_Y Z^N) \\ &\quad - \bar{\nabla}_Y (\bar{\nabla}_X Z_T + \bar{\nabla}_X Z^N) \\ &\quad - \bar{\nabla}_{[X, Y]} (Z_T + Z^N) \\ &= \bar{\nabla}_X \begin{pmatrix} \nabla_Y Z_T + h(Y, Z_T) \\ -A_{Z^N} Y + \nabla_Y^\perp Z^N \end{pmatrix} \\ &\quad - \bar{\nabla}_Y \begin{pmatrix} \nabla_X Z_T + h(X, Z_T) \\ -A_{Z^N} X + \nabla_X^\perp Z^N \end{pmatrix} \\ &\quad - \nabla_{[X, Y]} Z_T - h([X, Y], Z_T) \\ &\quad + A_{Z^N} [X, Y] + \nabla_{[X, Y]}^\perp Z^N, \end{aligned} \quad (2.15)$$

for all vector fields X, Y in TM and Z in $\overline{TM}$. If we use (2.12) in (2.15), we find that,

$$\begin{aligned} \bar{R}(X, Y)Z &= \bar{\nabla}_X (\lambda Y + h(Y, Z_T) + \nabla_Y^\perp Z^N) \\ &\quad - \bar{\nabla}_Y (\lambda X + h(X, Z_T) + \nabla_X^\perp Z^N) \\ &\quad - \lambda [X, Y] - h([X, Y], Z_T) + \nabla_{[X, Y]}^\perp Z^N \\ &= \nabla_X \lambda Y + h(X, \lambda Y) \\ &\quad - A_{h(Y, Z_T)} X + \nabla_X^\perp h(Y, Z_T) \\ &\quad - A_{\nabla_Y^\perp Z^N} X + \nabla_X^\perp \nabla_Y^\perp Z^N \\ &\quad - \nabla_Y \lambda X - h(Y, \lambda X) \\ &\quad + A_{h(X, Z_T)} Y - \nabla_Y^\perp h(X, Z_T) \\ &\quad + A_{\nabla_X^\perp Z^N} Y - \nabla_Y^\perp \nabla_X^\perp Z^N \\ &\quad - \lambda [X, Y] - h([X, Y], Z_T) \\ &\quad - \nabla_{[X, Y]}^\perp Z^N. \end{aligned} \quad (2.16)$$

From this last equation, we obtain that,

$$\begin{aligned} \bar{R}(X, Y)Z &= R^\perp(X, Y)Z^N - A_{h(Y, Z_T)} X \\ &\quad + A_{h(X, Z_T)} Y - h([X, Y], Z_T) \\ &\quad + \nabla_X^\perp h(Y, Z_T) - \nabla_Y^\perp h(X, Z_T). \end{aligned} \quad (2.16)$$

If we take ξ instead of X and we use (2.5) then we get

$$\begin{aligned} \alpha^2 \{-g(Y, Z)\xi + \eta(Z)Y\} &= R^\perp(\xi, Y)Z^N \\ &\quad - A_{h(Y, Z_T)} \xi \\ &\quad - h([X, Y], Z_T) \\ &\quad + \nabla_\xi^\perp h(Y, Z_T) \end{aligned} \quad (2.17)$$

If we take the inner product of (2.17) with $W \in \chi(M)$, then we find that

$$\begin{aligned} -\alpha^2 g(Y, Z)\eta(W) + \alpha^2 \eta(Z)g(Y, W) \\ = -g(A_{h(Y, Z_T)} \xi, W) \end{aligned} \quad (2.18)$$

If we use (2.8) in (2.18), then we get

$$\begin{aligned} \alpha^2 \{g(Y, W)\eta(Z) - g(Y, Z)\eta(W)\} \\ = -g(h(\xi, W), h(Y, Z_T)). \end{aligned} \quad (2.19)$$

From (2.14), we find that

$$\alpha^2 \{g(Y, W)\eta(Z) - g(Y, Z)\eta(W)\} = 0. \quad (2.20)$$

If we take ξ instead of W , we find

$$\alpha^2 \{\eta(Y)\eta(Z) - g(Y, Z)\} = 0. \quad (2.21)$$

Since α is different from zero, we have

$$\eta(Z)\eta(Y) - g(Y, Z) = 0. \quad (2.22)$$

But, this is not possible. Then the proof is complete.

3. References

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