REPUBLIC OF TURKEY
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SPECTRAL PROPERTIES OF SOME DIFFERENTIAL OPERATORS ON TIME SCALES

Master Thesis

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Program: Analysis and Functions Theory

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SUMMARY

Spectral properties of some differential operators on time scales

In this thesis, we try to get some basic features of different types of operators on time scale. This thesis includes six chapters. In chapter 1, historical improvement of time scale theory is given. In chapter 2, some fundamental properties, some notions and theorems of time scale calculus are explained. Then, first and second canonic forms of Dirac system in classical spectral theory are expressed on time scales in chapter 3. The physical meaning of Dirac system is stated. Therefore, some important properties of Dirac system are given on a time scale for the first and second canonic forms are given. Furthermore, the asymptotics estimate of the eigenfunction for the first and second canonic forms of Dirac system are found. In chapter 4, impulsive diffusion equation on time scales is studied. Before expressing the main results, the physical meaning of impulsive diffusion equation and some properties of this equation in spectral theory are given. Then, the basic properties of impulsive diffusion equation are generalized to an arbitrary time scale. And, asymptotic estimate of the eigenfunction for impulsive diffusion equation on a time scale is constructed. Eventually, Bessel equation on time scale is considered in chapter 5. Its physical properties are defined. Then, its basic properties are expressed on time scale. Moreover, the asymptotic estimate of the eigenfunction for Bessel equation is given. In chapter 6, some fundamental conclusions about thesis are given.

Key Words: Time Scales, Impulsive Diffusion Equation, Dirac System, Bessel Equation.

ÖZET

Zaman skalası üzerinde bazı diferensiyel operatörlerin spektral özellikleri

Bu tezde, zaman skalası üzerinde farklı tip operatörlerin bazı temel özelliklerini elde etmeye çalıştık. Tez, 6 bölümden oluşmaktadır. Birinci bölümde, zaman skalası teorisinin tarihsel gelişimi verildi. İkinci bölümde, zaman skalası teorisinin bazı temel özellikleri, bazı kavram ve teoremleri açıklandı. Daha sonra üçüncü bölümde, klasik spektral teorideki birinci ve ikinci kanonik formlar zaman skalasında verildi. Dirac sisteminin fiziksel anlamı ifade edildi. Böylece, zaman skalası üzerinde Dirac sisteminin birinci ve ikinci kanonik formu için bazı önemli özellikler verildi. Buna ilaveten, Dirac sisteminin birinci ve ikinci kanonik formuna ait öz fonksiyonların asimptotik ifadeleri elde edildi. Dördüncü bölümde, zaman skalası üzerinde impulsive difüzyon denklemi çalışıldı. Temel sonuçları ifade etmeden önce, impulsive difüzyone denkleminin spektral teorideki özellikleri ve fiziksel anlamı verildi. Daha sonra, impulsive difüzyon denkleminin bazı temel özellikleri keyfi zaman skalasına genelleştirildi. Zaman skalası üzerinde impulsive difüzyon denkleminin öz fonksiyonlarının asimptotik ifadesi oluşturuldu. Son olarak, beşinci bölümde bessel denklemi zaman skalası üzerinde ele alındı. Fiziksel özellikleri tanımlandı. Daha sonra zaman skalası üzerinde bu denklemin tüm temel özellikleri ifade edildi. Buna ilaveten, Bessel denkleminin öz fonksiyonlarına ait asimptotik ifadeler zaman skalası üzerinde verildi. Altıncı bölümde, tez ile ilgili bazı temel sonuçlar verildi.

Anahtar Kelimeler: Zaman Skalası, Impulsive Difüzyon denklemi, Dirac Sistemi, Bessel Denklemi.

SYMBOLS

$\mathbb{N}$	:	Natural Numbers
$\mathbb{R}$	:	Real Numbers
$\mathbb{Z}$	:	Integers
$\mathbb{C}$	:	Complex Numbers
$\mathbb{Q}$	:	Rational Numbers
σ	:	Forward jump operator
ρ	:	Backward jump operator
μ	:	Graininess function
Δf	:	Forward difference operator
f^{Δ}	:	Hilger (Delta) derivative
$f^{\Delta\Delta}$	:	Second order Hilger (Delta) derivative
$f^{\Delta^{\sigma}}$	:	Forward jump operator of first order Hilger derivative
$\mathbb{C}_{rd}(\mathbb{T})$	:	The set of all rd-continuos functions on $\mathbb{T}$
$\mathbb{C}^{1}_{rd}$	:	The set of all rd-continuos functions that are once differentiable
$\mathbb{C}^{2}_{rd}$	:	The set of all rd-continuos functions that are twice differentiable
$\mathbb{C}_h$	:	Hilger's complex plane
$\mathbb{R}_h$	:	Hilger's real axis
$\mathbb{I}_h$	:	Hilger's Imaginary circle
Re_h	:	Real part
ξ_h	:	Cylinder transformation
ξ_h^{-1}	:	Inverse cylinder transformation
$\mathcal{R}$	:	The collection of all regressive functions
$e_p(t,s)$	:	Exponential function on time scales
$v(t)$	:	Backward graininess function

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1 INTRODUCTION

Although analogous opinions have been obtained before and go back at least to the beginning of Riemann-Stieltjes integral which unifies sums and integrals, time scale theory was first presented by Stefan Hilger [1] in his doctoral dissertation under supervision of Bernard Aulbach in 1988 [2], [3]. The theory of time scale is a unification of the classical calculus of the functions with domain $\mathbb{R}$ and the discrete calculus defined on $\mathbb{Z}$ which extend these notions to more exotic domains. The time scale theory has many applications containing non-continuous domains, such as modeling of certain bug populations, chemical reactions, phytoremediation of metals, wound healing, maximization problems in economics and traffic problems. The most three familiar examples of calculus on time scales are differential, difference and quantum calculus. Recently, several authors have obtained many important results about different topics on time scales (see [4], [5], [6], [7], [8], [9], [10], [11]).

A small number of studies has been done about boundary value problems (BVP's) on time scales in literature. Elementary studies on these type problems for linear $\Delta-$differential equations were fulfilled by Chyan et al in 1998 and Agarwal, Bohner and Wong in 1999. In [12], the theory of positive operators according to a cone in a Banach space is applied to eigenvalue problems related to the second order linear $\Delta-$differential equations on time scales to prove existence of a smallest positive eigenvalue and then a theorem is proved to compare the smallest positive eigenvalue for two problems of that type. In [13], an oscillation theorem is given for Sturm-Liouville (SL) eigenvalue problem on time scales with separated boundary conditions and Rayleigh's principle is proved. In 2002, Agarwal, Bohner and O'Regan [14] presented some original existence results for time scale BVP's on an infinite interval. Guseinov [15] investigated some eigenfunction expansions for a SL eigenvalue problem on time scales in 2007. In this study, the existence of the eigenvalues and eigenfunctions is examined and mean square convergent and uniformly convergent expansions for eigenfunctions are expressed by Guseinov. Later, Huseynov and Bairamov generalized the conclusions of Guseinov to the more general eigenvalue problems in 2009 [16]. Zhang and Ma investigated solvability of SL problems on time scales at resonance in 2010 [17]. Erbe, Mert and Peterson [18] derived formulas to find two linearly independent solutions of SL dynamic equation in 2012. Tuna [19] considered dissipative SL operators in the limit-circle case on time scales in 2014. In 2016, Gulsen and Yilmaz explained some properties of Dirac system on time scales with separated boundary conditions [20]. There are many studies about spectral theory on time scales except those (see [21], [22]).

As it can be seen from literature, the studies about spectral theory on time scales have focused on SL equation. According to our investigation, there isn't any study about spectral analysis of impulsive diffusion equation and Bessel equation on time scales. So, this study will be the first with regard to spectral theory of impulsive diffusion equation and Bessel equation on time scales. Our study will be an important research for mathematicians to solve some direct and inverse spectral problems for any types of differential operators on time scales. Many subjects in spectral theory could be extended to the different areas. Classical results and theorems in spectral theory will be a special case of our results and theorems. Because, time scale theory generalize many concepts in classical calculus to an arbitrary time scale. For instance, classical derivate is defined as Hilger's derivative on time scale. Therefore, one can compute the derivative of a function in discrete case. Similarly, we can say similar things for integration. One can easily obtain the result of integral for a function in discrete case. This allows us to interpret the concepts differently. We are sure that these studies will lead to the creation of new areas in mathematics.

2 PRELIMINARIES OF TIME SCALE CALCULUS

Definition 2.1 [4] A time scale is known as an arbitrary, nonempty, closed subset of $\mathbb{R}$.

Common time scales are $\mathbb{R}$, $\mathbb{Z}$, $\mathbb{N}$ and $\mathbb{N}_0$. But also $[1,3] \cup [5.5,6]$, $[1,3] \cup \mathbb{N}$ and the Cantor set are time scales. Conversely, the sets $\mathbb{Q}$, $\mathbb{R} \setminus \mathbb{Q}$, $\mathbb{C}$ and $(1,3)$ are not time scales. One can use the symbol $\mathbb{T}$ to denote time scale. Throghout this study, we will use this symbol to denote an arbitrary time scale. A time scale of the form of a union of disjoint closed real intervals constitutes a good background for the study of population models. Such models arise, for instance, when a plant population exhibits exponential growth during the months of spring and summer, and at the beginning of autumn all plants die while the seeds remain in the ground. Now, let us remind some notions and theorems on time scale calculus.

Definition 2.2 [4] The forward jump operator $\sigma : \mathbb{T} \longrightarrow \mathbb{T}$ is defined by

$$\sigma(t) = \inf \{s \in \mathbb{T} : s > t\},$$

when the backward jump operator $\rho : \mathbb{T} \longrightarrow \mathbb{T}$ is expressed by

$$\rho(t) = \sup \{s \in \mathbb{T} : s < t\},$$

for $t \in \mathbb{T}$. Here, we set $\inf \phi = \sup \mathbb{T}$ (i,e., $\sigma(t) = t$ if $\mathbb{T}$ has maximum t) and $\sup \phi = \inf \mathbb{T}$ (*i.e.*, $\rho(t) = t$ if $\mathbb{T}$ has minimum t), where ϕ indicates the empty set.

Definition 2.3 [4] Let $t \in \mathbb{T}$. Then t is said to be

(i) right -scattered if $\sigma(t) > t$,

(ii) left-scattered if $\rho(t) < t$,

(iii) isolated if t is right-scattered and left-scattered,

(iv) right-dense if $t < \sup \mathbb{T}$ and $\sigma(t) = t$,

(v) left-dense if $t > \inf \mathbb{T}$ and $\rho(t) = t$,

(vi) dense if t is right-dense and left- dense.

Definition 2.4 [4] The forward graininess function μ: $\mathbb{T} \longrightarrow [0, \infty)$ is defined by

$$\mu(t) := \sigma(t) - t,$$

where $t \in \mathbb{T}$.

Definition 2.5 [4] The backward graininess function $v : \mathbb{T} \longrightarrow [0, \infty]$ is defined by $v(t) = t - \rho(t)$ where $t \in \mathbb{T}$.

Example 2.1. [4]

(i) If $\mathbb{T}=\mathbb{R}$, then

$$\sigma(t)=\inf\{s\in\mathbb{R}:s>t\}=\inf(t,\infty)=t,$$

and

$$\rho(t)=\sup\{s\in\mathbb{R}:s<t\}=\sup(-\infty,t)=t,$$

for any $t\in\mathbb{R}$. Thus, each point $t\in\mathbb{R}$ is dense. We can deduce that the graininess function has the form $\mu(t)\equiv 0$ for all $t\in\mathbb{R}$.

(ii) If $\mathbb{T}=\mathbb{Z}$, then one can easily obtain

$$\sigma(t)=\inf\{s\in\mathbb{Z}:s>t\}=\inf\{t+1,t+2,t+3,...\}=t+1,$$

and

$$\rho(t)=\sup\{s\in\mathbb{Z}:s<t\}=\sup\{t-1,t-2,t-3,...\}=t-1,$$

for any $t\in\mathbb{Z}$. Thus, every point $t\in\mathbb{Z}$ is isolated. In this case, the graininess function is $\mu(t)\equiv 1$ for all $t\in\mathbb{Z}$.

(iii) Let $h>0$ and if $\mathbb{T}=h\mathbb{Z}=\{hk:k\in\mathbb{Z}\}$, then for any $t\in h\mathbb{Z}$

$$\sigma(t)=\inf\{s\in h\mathbb{Z}:s>t\}=\inf\{t+nh:n\in\mathbb{N}\}=t+h,$$

and

$$\rho(t)=\sup\{s\in h\mathbb{Z}:s<t\}=\sup\{t-nh:n\in\mathbb{N}\}=t-h.$$

Hence every point $t\in h\mathbb{Z}$ is isolated and $\mu(t)=h$ for all $t\in h\mathbb{Z}$ and so again μ is constant.

(iv) Let $q>1$ and $q^{\mathbb{Z}}:=\left\{q^k:k\in\mathbb{Z}\right\}$ and $q^{\overline{\mathbb{Z}}}:=q^{\mathbb{Z}}\cup\{0\}$. Now let $\mathbb{T}=q^{\mathbb{Z}}$. In this case, we can deduce

$$\sigma(t)=\inf\{s\in\mathbb{T}:s>t\}=\inf\{q^n:n\in[m+1,\infty)\}=q^{m+1}=qq^m=qt,$$

and trivially $\sigma(0)=0$,where $t=q^m\in\mathbb{T}$. Furthermore

$$\begin{aligned}\rho(t) &= \sup\{s\in\mathbb{T}:s<t\}=\sup\{q^n:n\in[-\infty,m-1)\}\\ &= q^{m-1}=\frac{q^m}{q}=\frac{t}{q}.\end{aligned}$$

Hence every $t\in\mathbb{T}$ is isolated since $q>1$. Finally $\mu(t)=(q-1)t$ for all $t\in q^{\mathbb{Z}}$, i.e., in this case $\mu(t)$ is not constant.

Example 2.2. [4] If $\mathbb{T} = [-4,1] \cup \mathbb{N}$, then

$$\sigma(t) = \begin{cases} t & \text{if } t \in [-4,1] \\ t+1 & \text{otherwise} \end{cases},$$

while

$$\rho(t) = \begin{cases} t & \text{if } t \in [-4,1] \\ t-1 & \text{otherwise} \end{cases}.$$

Moreover

$$\mu(t) = \begin{cases} 0 & \text{if } t \in [-4,1] \\ 1 & \text{otherwise} \end{cases},$$

and

$$v(t) = \begin{cases} 0 & \text{if } t \in [-4,1] \\ 1 & \text{otherwise} \end{cases}.$$

We can summarize graininess function, forward and backward jump operators of some known time scales by using below table.

$\mathbb{T}$	$\mu(t)$	$\sigma(t)$	$\rho(t)$
$\mathbb{R}$	0	t	t
$\mathbb{Z}$	1	$t+1$	$t-1$
$h\mathbb{Z}$	h	$t+h$	$t-h$
$q^{\mathbb{N}}$	$(q-1)t$	qt	$\frac{t}{q}$
2^n	t	$2t$	$\frac{t}{2}$
$\mathbb{N}_0^2$	$2\sqrt{t}+1$	$\left(\sqrt{t}+1\right)^2$	$\left(\sqrt{t}-1\right)^2$

Table.1

Remark 2.1 [4] Note that $\sigma(t)$ and $\rho(t)$ are in $\mathbb{T}$ when $t \in \mathbb{T}$. Later on, we should define the set $\mathbb{T}^k$ which is given by the following way:

$$\mathbb{T}^k = \begin{cases} \mathbb{T}\backslash\left(\rho(\sup\mathbb{T}), \sup\mathbb{T}\right], & if\ \sup\mathbb{T} < \infty, \\ \mathbb{T}, & if\ \sup\mathbb{T} = \infty. \end{cases}$$

Furthermore, f^{σ}:$\mathbb{T} \longrightarrow \mathbb{R}$ is defined by $f^{\sigma}(t) = f(\sigma(t))$ for all $t \in \mathbb{T}$, i.e. $f^{\sigma} = f \circ \sigma$, where f:$\mathbb{T} \longrightarrow \mathbb{R}$ is a function.

Example 2.3. [4] If $\mathbb{T} = [0,1] \cup \{3,5\} \cup [6,7] \cup \{8\}$, let us find $\mathbb{T}^k$. Since $\sup\mathbb{T} = 8 < \infty$, $\mathbb{T}^k = \mathbb{T} - \sup\mathbb{T} = \mathbb{T} - \{8\} = [0,1] \cup \{3,5\} \cup [6,7]$.

Next we consider f:$\mathbb{T} \longrightarrow \mathbb{R}$ and define the Δ-derivative (or Hilger- Derivative) of f at $t \in \mathbb{T}^k$.

Definition 2.6 [4] Suppose that f:$\mathbb{T} \longrightarrow \mathbb{R}$ is a function and $t \in \mathbb{T}^k$. Then, $f^{\Delta}(t)$ is defined as the number (provided that it exists) with the attribute that for every $\varepsilon > 0$, there exists a neighborhood U of t i.e., $U = (t-\delta, t+\delta) \cap \mathbb{T}$ for some $\delta > 0$) such that

$$\left|[f(\sigma(t)) - f(s)] - f^{\Delta}(t)[\sigma(t) - s]\right| \leq \varepsilon |\sigma(t) - s|,$$

for all $s \in U$. We call this number $f^{\Delta}(t)$ the Δ derivative of f at t.

Theorem 2.1 [4] Suppose $f : \mathbb{T} \longrightarrow \mathbb{R}$ is a function and $t \in \mathbb{T}^k$. At that time, the following holds:
(i) If f is differentiable at t, then f is continuous at t.
(ii) If f is continuous at t and t is right-scattered, then f is differentiable at t with

$$f^{\Delta}(t) = \frac{f(\sigma(t)) - f(t)}{\mu(t)}.$$

(iii) If t is right-dense, then f is differentiable at t iff

$$\lim_{s \longrightarrow t} \frac{f(t) - f(s)}{t-s},$$

exists as a finite number. If this is the case, then

$$f^{\Delta}(t) = \lim_{s \longrightarrow t} \frac{f(t) - f(s)}{t-s}.$$

(iv) If f is differentiable at t, then

$$f(\sigma(t)) = f(t) + \mu(t) f^{\Delta}(t).$$

(v) Finally, if $f : \mathbb{T} \longrightarrow \mathbb{R}$ is differentiable, then $f^{\sigma} : \mathbb{T}^k \longrightarrow \mathbb{R}$ is also differentiable and

$$f^{\sigma\Delta} = \sigma^{\Delta} f^{\Delta\sigma}.$$

Example 2.4. [4]
(i) If $\mathbb{T} = \mathbb{R}$, then we can deduce from Theorem 2.1 (iii) that f:$\mathbb{R} \longrightarrow \mathbb{R}$ is delta differentiable at $t \in \mathbb{R}$ iff

$$f'(t) = \lim_{s \longrightarrow t} \frac{f(t) - f(s)}{t-s},$$

exists, i.e., iff f is differentiable (in the usual sense) at t. Then, we get

$$f^{\Delta}(t) = \lim_{s \to t} \frac{f(t) - f(s)}{t-s} = f'(t).$$

(ii) If $\mathbb{T} = \mathbb{Z}$, then Theorem 2.1 (ii) yields that $f : \mathbb{Z} \longrightarrow \mathbb{R}$ is Δ differentiable at $t \in \mathbb{Z}$ with

$$f^{\Delta}(t) = \frac{f(\sigma(t)) - f(t)}{\mu(t)} = \frac{f(t+1) - f(t)}{1} = f(t+1) - f(t) = \Delta f(t),$$

where Δ is the well-known difference operator defined like above.

Theorem 2.2 [4] Suppose $f, g : \mathbb{T} \longrightarrow \mathbb{R}$ are differentiable at $t \in \mathbb{T}^k$. Then
(i) $f + g : \mathbb{T} \longrightarrow \mathbb{R}$ is differentiable at t with the delta derivative

$$(f+g)^{\Delta}(t) = f^{\Delta}(t) + g^{\Delta}(t).$$

(ii) For every constant α, $\alpha f : \mathbb{T} \longrightarrow \mathbb{R}$ is differentiable at t with the delta derivative

$$(\alpha f)^{\Delta}(t) = \alpha f^{\Delta}(t).$$

(iii) $fg : \mathbb{T} \longrightarrow \mathbb{R}$ is differentiable with the delta derivative

$$(fg)^{\Delta}(t) = f^{\Delta}(t)\, g(t) + f(\sigma(t))\, g^{\Delta}(t) = f(t)\, g^{\Delta}(t) + f^{\Delta}(t)\, g(\sigma(t)).$$

(iv) If $f(t)\, f(\sigma(t)) \neq 0$, then $\dfrac{1}{f}$ is differentiable at t with delta-derivative

$$\left(\frac{1}{f}\right)^{\Delta}(t) = -\frac{f^{\Delta}(t)}{f(t)\ f(\sigma(t))}.$$

(v) If $g(t)\, g(\sigma(t)) \neq 0$, then $\dfrac{f}{g}$ is differentiable with delta-derivative.

$$\left(\frac{f}{g}\right)^{\Delta}(t) = \frac{f^{\Delta}(t)\, g(t) - f(t)\, g^{\Delta}(t)}{g(t)\, g(\sigma(t))}.$$

Theorem 2.3 (Chain Rule) [4] Let $f : \mathbb{R} \longrightarrow \mathbb{R}$ be continuously differentiable and assume $g : \mathbb{T} \longrightarrow \mathbb{R}$ is Δ differentiable. Hence, $f \circ g : \mathbb{T} \longrightarrow \mathbb{R}$ is Δ differentiable and the equality

$$(f \circ g)^{\Delta}(t) = \left\{ \int_0^1 f'\left(g(t) + h\mu(t)\, g^{\Delta}(t)\right) dh \right\} g^{\Delta}(t),$$

holds.

Example 2.5. [4] Define $g : \mathbb{Z} \to \mathbb{R}$ and $f : \mathbb{R} \to \mathbb{R}$ by $g(t) = 2t$ and $f(t) = t^2$. Then

$$g^{\Delta}(t) = g(t+1) - g(t) = 2(t+1) - 2t = 2$$

and $f'(t) = 2t$,

$$\begin{aligned} (f \circ g)^{\Delta}(t) &= \left\{ \int_0^1 f' g(t) + h\mu(t)\, g^{\Delta}(t)) dh \right\} g^{\Delta}(t) \\ &= 2\left\{ \int_0^1 f'(2t + 2h)\, dh \right\} \\ &= 8t + 4. \end{aligned}$$

On the other hand, we obtain

$$\Delta f\left(g\left(t\right)\right)=f\left(g\left(t+1\right)\right)-f\left(g\left(t\right)\right)=8t+4.$$

Definition 2.7. [4] $f:\mathbb{T}\rightarrow\mathbb{R}$ is a regulated function if its right sided limits exist (finite) at all right dense points on $\mathbb{T}$ and its left sided limits exist (finite) at all left dense points on $\mathbb{T}$.

Example 2.6. [4] Consider the time scale $\mathbb{T}=\{0\}\cup\left\{\frac{1}{n}:n\in\mathbb{N}\right\}\cup\{2\}\cup\left\{2-\frac{1}{n}:n\in\mathbb{N}\right\}$. Then, $f:\mathbb{T}\rightarrow\mathbb{R}$ which is defined by

$$f(t)=\begin{cases} t, & if\ t\neq 2 \\ 0, & if\ t=2 \end{cases},$$

is regulated.

Continuity of a function at a point $t\in\mathbb{T}$ depends on the appearance of t as" right-dense" $(t=\sigma\left(t\right))$ or "left-dense" $(t=\rho\left(t\right))$. Thus, for any $t\in\mathbb{T}$, a right-dense-continuous (usually written as $rd-$ continuous function is defined as follows.

Definition 2.8 [4] **(The right-dense continuity)** Assume $f:\mathbb{T}\rightarrow\mathbb{R}^{n}$. We define f as right-dense continuous or $rd-$continuous if

$$\lim_{s\rightarrow t^{+}}f\left(s\right)=f\left(t\right),$$

where t is right-dense and

$$\lim_{s\rightarrow t}f\left(s\right),$$

exists and is finite for all $t\in\mathbb{T}$, where t is left-dense. The set of all $rd-$continuous functions on $\mathbb{T}$ is indicated by $C_{rd}\left(\mathbb{T};\mathbb{R}^{n}\right)$.

Next definition due to Hilger describes the so-called right-Hilger-continuous function $f\left(t,x\right)$ where the ordered $n-$pair $(t,x)\in\mathbb{T}\times\mathbb{R}^{n}$. This is a more generalised definition and we have introduced this particular term for functions of several variables to avoid confusion with $rd-$continuous functions of one variable.

Definition 2.9 (The right-Hilger-continuous) [4] Consider an arbitrary time scale $\mathbb{T}$. A function $f:\mathbb{T}^{k}\times\mathbb{R}^{n}\rightarrow\mathbb{R}$ having the property that f is continuous at each (t,x) where t is right-dense; and the limits

$$\lim_{(s,y)\rightarrow(t^{-},x)}f\left(s,y\right)\ \text{and}\ \lim_{y\rightarrow x}f\left(t,y\right)$$

both exist and are finite at each (t,x), where t is left-dense, is said to be right-Hilger-continuous on $\mathbb{T}^{k}\times\mathbb{R}^{n}$.

Remark 2.2. [4] It should be noted that we will write f defined above as $rd-$continuous if it is a function of t only, that is, $f(t,x) = g(t)$ for all $t \in \mathbb{T}$.

Remark 2.3. [4] It can also be seen that a right-Hilger-continuous function f will be continuous if $f(t,x) = h(x)$ for all $t \in \mathbb{T}$.

Definition 2.10 (The left-dense continuity) [4] Let $\mathbb{T}$ be an arbitrary time scale. f: $\mathbb{T} \to \mathbb{R}$ is said to be ld-continuous if it is continuous at each $t \in \mathbb{T}$ that is left-dense and $\lim\limits_{s \to t^+} f(s)$ exists and is finite at each $t \in \mathbb{T}$ that is right dense.

As for right-Hilger-continuous functions, the term 'left-Hilger-continuous' is used in equivalence with the term 'ld-continous' for a function of two or more variables, the first of which should be from an arbitrary time scale.

Definition 2.11 (The left-Hilger-continuity) [4] Let $\mathbb{T}$ be an arbitrary time scale. A mapping $f : [a,b]_{k,\mathbb{T}} \times \mathbb{R} \to \mathbb{R}$ is called left-Hilger-continuous if f is continuous at each (t,x) where t is left-dense; and the limits

$$\lim_{(s,y) \to (t^+,x)} f(s,y) \text{ and } \lim_{y \to x} f(t,y)$$

both exist and are finite at each (t,x), where t is right-dense.

Remark 2.4. [4] The above two remarks are also hold for left-Hilger-continuous and $ld-$continuous.

Example 2.7. [4] The following functions are left-Hilger-continuous:

1. $f(t,p) = tp^2$, where $t \in \mathbb{T}_k$ and $p \in \mathbb{R}$. Note that the composition function $t(x(t))^2$ will be ld-continuous on $\mathbb{T}_k \times \mathbb{R}$. Therefore, by definition, f left-Hilger-continuous on $\mathbb{T}_k \times \mathbb{R}$;

2. $f(t,p) = t^2 \ln p$, where $t \in \mathbb{T}_k$ and $p \in [0,k]$ where $k > 0$ is a continuous function. Note that the composition function $t^2 \ln x(t)$ will be ld-continuous on $\mathbb{T}_k$. Therefore, $f(t,p)$ is left-Hilger-continuous for all $(t,p) \in \mathbb{T}_k \times [0,k]$;

3. $f(t,p) = \frac{1}{1+t} e^p$, where $t \in (-1,0]_{\mathbb{T}}$ and $p \in [0,\infty)$. Note that the composition function $\frac{1}{1+t} ex[x(t)]$ will be ld-continuous for all $t \in (-1,0]_{k,\mathbb{T}}$. Therefore, our f is left-Hilger-continous on $(-1,0]_{k,\mathbb{T}} \times [0,\infty)$;

4. $f(t,p) = \rho(t) + p$, where $t \in [0,1]_{\mathbb{T}}$ and $p \in \mathbb{R}$. Then the composition function $\rho(t) + x(t)$ will be ld-continuous for all $t \in [0,1]_{k,\mathbb{T}}$. Therefore, our f is left-Hilger-continuous on $[0,1]_{k,\mathbb{T}} \times \mathbb{R}$.

Example 2.8. [4] Let $\mathbb{T} = \mathbb{N}_0 \cup \left\{1 - \frac{1}{n} : n \in \mathbb{N}\right\}$ and define $f : \mathbb{T} \to \mathbb{R}$ by

$$f(t) = \begin{cases} 0, & if\ t \in \mathbb{N} \\ t, & \text{otherwise} \end{cases}.$$

It is easy to notice that f is continuous at the isolated points. So, the points that need our careful attention are the right-scattered point 0 and the left-dense point 1. The right-sided limit of f at 0 exists and is finite (is equal to 0). The left-sided limit of f at 1 exists and is equal to 1 (finite). So, f is rd-continuous on $\mathbb{T}$.

Remark 2.5. [4]

(i) If f is continuous, then f is rd-continuous.

(ii) σ is rd-continuous.

(iii) μ is rd-continuous.

(iv) The delta-derivative of the forward jump operator σ^{Δ} is rd-continuous.

Definition 2.12 [4] The definite Cauchy integral of a function f is defined by

$$\int_r^s f(t)\,\Delta t = F(s) - F(r),$$

for all $r, s \in \mathbb{T}$. $F : \mathbb{T} \to \mathbb{R}$ is called antiderivative of $f : \mathbb{T} \to \mathbb{R}$ if $F^{\Delta}(t) = f(t)$ for all $t \in \mathbb{T}^k$.

Example 2.9. [4] If $\mathbb{T} = \mathbb{Z}$, find the indefinite Δ integral $\int a^t \Delta t$, where $a \neq 1$ is a constant.
Solution: Since $\left(\frac{a^t}{a-1}\right)^{\Delta} = \Delta\left(\frac{a^t}{a-1}\right) = a^t$,we can easily get

$$\int a^t \Delta t = \frac{a^t}{a-1} + c,$$

where c is an arbitrary constant.

Theorem 2.4 (Existence of Antiderivative) [4] Every rd-continuous function possesses an antiderivative. Furthermore if $t_0 \in \mathbb{T}$, then F defined by

$$F(t) := \int_{t_0}^t f(\tau)\,\Delta\tau,$$

for $t \in \mathbb{T}$ is an antiderivative of f.

Theorem 2.5. [4] Let $a, b \in \mathbb{T}$ and $f \in C_{rd}$.

(i) If $\mathbb{T} = \mathbb{R}$, then

$$\int_a^b f(t)\,\Delta t = \int_a^b f(t)\,dt,$$

where the integral on the right side is classical Riemann type integral.

(ii) If $[a,b]$ has only isolated points, then

$$\int_a^b f(t)\,\Delta t = \begin{cases} \sum_{t\in[a,b)} \mu(t) f(t), & \text{if} \quad a<b \\ 0, & \text{if} \quad a=b \\ -\sum_{t\in[b,a)} \mu(t) f(t), & \text{if} \quad a>b. \end{cases}$$

(iii) If $\mathbb{T} = h\mathbb{Z} = \{hk : k \in \mathbb{Z}\}$, where $h>0$, then

$$\int_a^b f(t)\,\Delta t = \begin{cases} \sum_{k=\frac{a}{h}}^{\frac{b}{h}-1} f(kh)\,h, & \text{if} \quad a<b \\ 0, & \text{if} \quad a=b \\ -\sum_{k=\frac{b}{h}}^{\frac{a}{h}-1} f(kh)\,h, & \text{if} \quad a>b \end{cases}$$

(iv) If $\mathbb{T} = \mathbb{Z}$, then

$$\int_a^b f(t)\,\Delta t = \begin{cases} \sum_{t=a}^{b-1} f(kh)\,h, & \text{if} \quad a<b \\ 0, & \text{if} \quad a=b \\ -\sum_{t=b}^{a-1} f(kh)\,h, & \text{if} \quad a>b \end{cases}$$

Theorem 2.6 [4] If $a,b,c \in \mathbb{T}, \alpha \in \mathbb{R}$ and $f,g \in C_{rd}$ then

(i) $\int_a^b [f(t)+g(t)]\,\Delta t = \int_a^b f(t)\,\Delta t + \int_a^b g(t)\,\Delta t.$

(ii) $\int_a^b (\alpha f)(t) = \alpha \int_a^b f(t)\,\Delta t.$

(iii) $\int_a^b f(t)\,\Delta t = -\int_b^a f(t)\,\Delta t.$

(iv) $\int_a^b f(t)\,\Delta t = \int_a^c f(t)\,\Delta t + \int_c^b f(t)\,\Delta t.$

(v) $\int_a^a f(t)\,\Delta t = 0.$

(vi) If $f(t) \geq 0$ for all $a \leq t < b$, then $\int_a^b f(t) \geq 0.$

Here, you can find some basic properties of the time scales $\mathbb{R}$ and $\mathbb{Z}$ in just one table.

Time Scale ($\mathbb{T}$)	$\mathbb{R}$	$\mathbb{Z}$
$\sigma(t)$	t	$t+1$
$\rho(t)$	t	$t-1$
$\mu(t)$	0	1
$f^{\Delta}(t)$	$f'(t)$	$\Delta f(t)$
$\int_a^b f(t)\,\Delta t$	$\int_a^b f(t)\,dt$	$\sum_{t=a}^{b-1} f(t)$ $(if\ a<b)$

Table.2

Definition 2.13 [4] Let $\alpha \in \mathbb{T}$, $\sup \mathbb{T} = \infty$ and f be rd-continuous on $[a, \infty)$. We denote the improper integral by

$$\int_a^\infty f(t)\,\Delta t := \lim_{b\to\infty} \int_a^b f(t)\,\Delta t$$

as long as this limit exists. If it exists, then the improper integral converges, whereas if it does not exist, then the improper integral diverges.

Example 2.10. Evaluate the integral

$$\int_1^\infty \frac{1}{t^2}\Delta t,$$

for $\mathbb{T} = q^{\mathbb{N}_0}$, where $q > 1$.

Solution: Since all points of the given time scale are isolated, we get

$$\int_1^\infty \frac{1}{t^2}\Delta t = \sum_{t=1}^{\infty} \frac{q-1}{t},$$

by the theorem 2.5. (b) where $\mu(t) = (q-1)t$.

Definition 2.14 [4] $p : \mathbb{T} \to \mathbb{R}$ is said to be regressive given that $1 + \mu(t)\, p(t) \neq 0$ for all $t \in \mathbb{T}^k$ holds. The set of all regressive and rd-continuous functions is indicated by $\mathcal{R} = \mathcal{R}(\mathbb{T}) = \mathcal{R}(\mathbb{T}, \mathbb{R})$.

Definition 2.15 [4] Let $p, q \in \mathcal{R}$. Define the "circle plus " addition $\oplus$ on $\mathcal{R}$ by

$$(p \oplus q)(t) := p(t) + q(t) + \mu(t)\, p(t)\, q(t),$$

for all $t \in \mathbb{T}^k$.

Definition 2.16 [4] The "circle minus" substraction $\ominus$ on $\mathcal{R}$ can be defined as

$$(p \ominus q)(t) := (p \oplus (\ominus q))(t),$$

for all $t \in \mathbb{T}^k$, where

$$(\ominus p)(t) := -\frac{p(t)}{1 + \mu(t)\, p(t)}.$$

Remark 2.6. [4] Let us consider the following circle minus substraction

$$(\ominus p)(t) := -\frac{p(t)}{1 + \mu(t)\, p(t)},$$

for all $t \in \mathbb{T}^k$. Also note that if $p, q \in \mathcal{R}$, then $p \oplus q$, $p \ominus q$ and $\ominus p \in \mathcal{R}$. Moreover if $p, q \in \mathcal{R}$, we can deduce the following equations using the definitions introduced above.

(i) $p \ominus p = 0$.

(ii) $\ominus(\ominus p) = p$.

(iii) $p \ominus q = \frac{p-q}{1+\mu q}$.

(iv) $\ominus(p \ominus q) = q \ominus p$

(v) $\ominus(p \oplus q) = (\ominus p) \oplus (\ominus q)$.

Definition 2.17 [4] For $h > 0$, the cylinder transformation $\xi_h : \mathbb{C}_h \to \mathbb{Z}_h$ is defined by

$$\xi_h(z) = \frac{1}{h} Log(1 + zh),$$

where Log is the principal logarithm function. If $h = 0$, we define $\xi_0(z) = z$ for all $z \in \mathbb{C}$.

Definition 2.18 [4] Let $p \in \mathcal{R}$. Then, the exponential function is defined by

$$e_p(t, s) = \exp\left(\int_s^t \xi_{h(\tau)}(p(\tau))\,\Delta\tau\right),$$

for $s, t \in \mathbb{T}$, where ξ_h is the cylinder transformation

Theorem 2.7 [4] If $p, q \in \mathcal{R}$, then

(i) $e_0(t, s) \equiv 1$ and $e_p(t, t) \equiv 1$;

(ii) $e_p(\sigma(t), s) = (1 + \mu(t) p(t)) e_p(t, s)$;

(iii) $\frac{1}{e_p(t,s)} = e_{\ominus p}(t, s)$;

(iv) $e_p(t, s) = \frac{1}{e_p(t,s)} = e_{\ominus p}(t, s)$;

(v) $e_p(t, s) e_p(s, r) = e_p(t, r)$;

(vi) $e_p(t, s) e_q(t, s) = e_{p \oplus q}(t, s)$;

(vii) $\frac{e_p(t,s)}{e_q(t,s)} = e_{p \ominus q}(t, s)$;

$(viii)$ $\left(\frac{1}{e_p(0,s)}\right)^{\Delta} = -\frac{p}{e_p^{\sigma}(0,s)}$.

To be able to formulate the next theorem, we introduce the following new definition.

Definition 2.19 [4] The set of all positively regressive elements of $\mathcal{R}$ is defined by

$$\mathcal{R}^+ = \mathcal{R}^+(\mathbb{T}, \mathbb{R}) = \{p \in \mathcal{R} : 1 + \mu(t) p(t) > 0 \text{ for all } t \in \mathbb{T}\}.$$

Remark 2.7 [4] If $p \in C_{rd}$ and $p(t) \geq 0$ for all $t \in \mathbb{T}$, then $p \in \mathcal{R}^+$.

Theorem 2.8 [4] If $p, q \in \mathcal{R}$,then

$$e_{p \ominus q}^{\Delta}(., t_0) = (p - q) \frac{e_p(., t_0)}{e_q^{\sigma}(., t_0)}.$$

Theorem 2.9 [4] If $p \in \mathcal{R}$ and $a, b, c \in \mathbb{T}$ then

$$[e_p(c,.)]^{\Delta} = -p\left[e_p(c,.)\right]^{\sigma},$$

and

$$\int_a^b p(t)\, e_p(c, \sigma(t))\, \Delta t = e_p(c,a) - e_p(c,b).$$

Theorem 2.10 (**Sign of the Exponential Function**) [4] Let $p \in \mathcal{R}$ and $t_0 \in \mathbb{T}$.
(i) If $p \in \mathcal{R}^{+}$, then $e_p(t\ , t_0) > 0$ for all $t \in \mathbb{T}$.
(ii) If $1 + \mu(t)\, p(t) < 0$ for some $t \in \mathbb{T}^k$, then $e_p(t, t_0)\, e_p(\sigma(t), t_0) < 0$.
(iii) If $1 + \mu(t)\, p(t) < 0$ for all $t \in \mathbb{T}^k$, then $e_p(t, t_0)$ changes sign at every point $t \in \mathbb{T}$.

Here, we can see the definition of some exponential functions on some known time scales.

$\mathbb{T}$	$e_\alpha(t, t_0)$
$\mathbb{R}$	$e^{\alpha(t-t_0)}$
$\mathbb{Z}$	$(1+\alpha)^{(t-t_0)}$
$h\mathbb{Z}$	$(1+\alpha h)^{\frac{(t-t_0)}{h}}$
$\frac{1}{n}\mathbb{Z}$	$\left(1+\frac{\alpha}{n}\right)^{n(t-t_0)}$
$q^{\mathbb{N}_0}$	$\Pi_{s\in[t_0,t)}\left[1+(q-1)\,\alpha s\right]$ if $t > t_0$
$2^{\mathbb{N}_0}$	$\Pi_{s\in[t_0,t)}\left(1+\alpha s\right)$ if $t > t_0$

Table.3

Definition 2.20. [4] If $p \in \mathcal{R}$, then the first order homogeneous linear dynamic equation $y^{\Delta} = p(t)\, y$ is called regressive.

Theorem 2.11. [4] Suppose that $y^{\Delta} = p(t)\, y$ is regressive and fix $t_0 \in \mathbb{T}$. Then, $e_p(., t_0)$ is a solution of the initial value problem (IVP)

$$y^{\Delta} = p(t)\, y,\ y(t_0) = 1 \tag{2.1}$$

on $\mathbb{T}$.

Theorem 2.12. [4] If $y^{\Delta} = p(t)\, y$ is regressive, then the only solution of the problem (2.1) is given by $e_p(., t_0)$.

Definition 2.21. [4] A first-order nonhomogeneous linear dynamic equation is of the form

$$y^{\Delta} = p(t)\, y + f(t)\,. \tag{2.2}$$

Theorem 2.13. [4] Suppose that $y^{\Delta} = p(t)\, y$ is regressive. Let $t_0 \in \mathbb{T}$ and $y_0 \in \mathbb{R}$. The unique solution of the IVP

$$y^{\Delta} = p(t)\, y,\ y(t_0) = y_0$$

is given by

$$y(t) = e_p(., t_0) y_0.$$

To find the solution of a first-order IVP ,we can use the so-called variation of constants or variation of parameters method.

Theorem 2.14. (**Variation of constants method**) [4] Suppose that $y^{\Delta} = p(t)\, y$ is regressive. Let $t_0 \in \mathbb{T}$ and $y_0 \in \mathbb{R}$. The unique solution of the IVP

$$y^{\Delta} = p(t)\, y + f(t)\,, y(t_0) = y_0$$

is given by

$$y(t) = e_p(., t_0) y_0 + \int_{t_0}^{t} e_p(t, \sigma(\tau)) f(\tau) \Delta\tau.$$

Definition 2.22. **(Trigonometric functions)** [4] If $p \in C_{rd}$ and $\mu p^2 \in \mathcal{R}$, then we define the the trigonometric functions $\cos_p$ and $\sin_p$ by

$$\cos_p = \frac{e_{ip} + e_{-ip}}{2} \quad \sin_p = \frac{e_{ip} - e_{-ip}}{2i} \quad .$$

Definition 2.23. **(Wronskian)** [4] For two differentiable functions y_1 and y_2, we define the Wronskian $W(y_1, y_2)$ by

$$W(y_1, y_2) = \det \begin{pmatrix} y_1(t) & y_2(t) \\ y_1^{\Delta}(t) & y_2^{\Delta}(t) \end{pmatrix}.$$

Definition 2.24 [4] Let us consider a special case of the corresponding homogeneous equation of (2.1), where $p(t)$ and $q(t)$ are constant, i.e., we look at an equation of the type

$$y^{\Delta\Delta} + \alpha y^{\Delta} + \beta y = 0, \tag{2.3}$$

with $\alpha, \beta \in \mathbb{R}$ on $\mathbb{T}$. Furthermore, we require that dynamic equation (2.3) is regressive, which means

$$1 - \alpha\mu(t) + \beta\mu^2(t) \neq 0,$$

for $t \in \mathbb{T}^K$, i.e., $\beta\mu - \alpha \in \mathcal{R}$. Next we try to find numbers $\lambda \in \mathbb{C}$ with $1 + \lambda\mu(t) \neq 0$ for $t \in \mathbb{T}^k$ such that

$$y(t) = e_\lambda(t, t_0),$$

satisfies (2.3). If $y(t) = e_\lambda(t, t_0)$,then

$$y^{\Delta\Delta}(t) + \alpha y^\Delta(t) + \beta y^\Delta(t) = \lambda^2 e_\lambda(t, t_0) + \alpha\lambda e_\lambda(t, t_0) + \beta e_\lambda(t, t_0) = \left(\lambda^2 + \alpha\lambda + \beta\right) e_\lambda(t, t_0).$$

Using the fact that $e_\lambda(t, t_0)$ is never zero, we can deduce that $y(t) = e_\lambda(t, t_0)$ is a solution of (2.3) iff λ is a solution of the characteristic equation

$$\lambda^2 + \alpha\lambda + \beta = 0. \tag{2.4}$$

The solutions of that characteristic equation are given by

$$\lambda_{1,2} = \frac{-\alpha \pm \sqrt{\alpha^2 - 4\beta}}{2}.$$

Remark 2.8 [4] If λ_1 and λ_2 are solutions of (2.4), then (2.3) is regressive iff $\lambda_1, \lambda_2 \in \mathcal{R}$.

Theorem 2.15 [4] Assume we have a regressive equation of the form (2.3) and we know that $\overline{y}$ is a particular solution of the corresponding nonhomogeneous equation. Let furthermore λ_1 and λ_2 be regressive solution of (2.4). Then the general solution of the nonhomogeneous equation has the form

$$y(t) = a_1 e_{\lambda_1} + a_2 e_{\lambda_2} + \overline{y}.$$

Definition 2.25 [4] Consider the below nonhomogeneous second order linear dynamic equation

$$y^{\Delta\Delta} + p(t) y^\Delta + q(t) y = g(t), \tag{2.5}$$

where we assume that $p, q, g \in C_{rd}$. If we introduce the operator $L_2 : C_{rd}^2 \to C_{rd}$ by

$$L_2 y(t) = y^{\Delta\Delta} + p(t) y^\Delta + q(t) y$$

for $t \in \mathbb{T}^{k^2}$. Then, (2.5) can be written as $L_2 y = g$. We try to find a particular solution of the IVP

$$L_2 y(t) = 0,\ y(t_0) = y_0,\ y^\Delta(t_0) = y_0^\Delta,$$

where $y_0, y_0^\Delta \in \mathbb{R}$. If y_1 and y_2 are the solutions of the $L_2 y(t) = 0$, then

$$y(t) = \alpha(t) y_1(t) + \beta(t) y_2(t),$$

is a solution of the $L_2 y(t) = 0$, where α and β are functions to be determined.

Definition 2.26 [4] Let us consider the nonhomogenous equation $L_2 y(t) = g(t)$, where

$$L_2 y(t) = y^{\Delta\Delta} + p(t) y^{\Delta} + q(t) y. \tag{2.6}$$

We try to find the particular solution of (2.6) of the form

$$y(t) = \alpha(t) y_1(t) + \beta(t) y_2(t).$$

Assume that such a y is a solution of (2.6). Note that

$$\begin{aligned} y^{\Delta}(t) &= \alpha^{\Delta}(t) y_1^{\sigma}(t) + \alpha(t) y_1^{\Delta}(t) + \beta^{\Delta}(t) y_2^{\sigma}(t) + \beta(t) y_2^{\Delta(t)} \\ & \alpha(t) y_1^{\Delta}(t) + \beta(t) y_2^{\Delta}(t), \end{aligned}$$

provided α and β satisfy

$$\alpha^{\Delta}(t) y_1^{\sigma}(t) + \beta^{\Delta}(t) y_2^{\sigma}(t) = 0. \tag{2.7}$$

Assuming that α and β can be picked so that (2.7) is satisfied, we proceed to calculate

$$y^{\Delta\Delta}(t) = \alpha^{\Delta}(t) y_1^{\Delta^{\sigma}}(t) + \alpha(t) y_1^{\Delta\Delta}(t) + \beta^{\Delta} y_2^{\Delta^{\sigma}}(t) + \beta(t) y_2^{\Delta\Delta}(t).$$

Hence

$$\begin{aligned} L_2 y(t) &= y^{\Delta\Delta}(t) + p(t) y^{\Delta}(t) + q(t) y(t) \\ &= \alpha(t) L_2 y_1(t) + \beta(t) L_2 y_2(t) + \alpha^{\Delta}(t) y_1^{\Delta^{\sigma}}(t) + \beta^{\Delta} y_2^{\Delta^{\sigma}}(t) \\ &= \alpha^{\Delta}(t) y_1^{\Delta^{\sigma}}(t) + \beta^{\Delta} y_2^{\Delta^{\sigma}}(t), \end{aligned}$$

since y_1 and y_2 solve (2.6). Therefore α and β need to satisfy the additional equation

$$\alpha^{\Delta}(t) y_1^{\Delta^{\sigma}}(t) + \beta^{\Delta} y_2^{\Delta^{\sigma}}(t) = g(t).$$

This equation, together with equation (2.7), leads to the following system of equations for α and β :

$$\begin{pmatrix} y_1^{\sigma}(t) & y_2^{\sigma}(t) \\ y_1^{\Delta^{\sigma}}(t) & y_2^{\Delta^{\sigma}}(t) \end{pmatrix} \begin{pmatrix} \alpha^{\Delta}(t) \\ \beta^{\Delta}(t) \end{pmatrix} = \begin{pmatrix} 0 \\ g(t) \end{pmatrix}.$$

We solve for α^{Δ} and β^{Δ} (recall that y_1 and y_2 for a fundamental system for 2.3.) to obtain

$$\begin{pmatrix} \alpha^{\Delta}(t) \\ \beta^{\Delta}(t) \end{pmatrix} = \frac{1}{W^{\sigma}(y_1, y_2)(t)} \left(\begin{array}{cc} y_2^{\Delta^{\sigma}}(t) & -y_2^{\sigma}(t) \\ -y_1^{\Delta^{\sigma}}(t) & y_1^{\sigma}(t) \end{array} \right) \begin{pmatrix} 0 \\ g(t) \end{pmatrix}$$

and therefore

$$\alpha^{\Delta}(t) = -\frac{y_2^{\sigma}(t)\, g(t)}{W^{\sigma}(y_1, y_2)(t)}, \quad \beta^{\Delta}(t) = \frac{y_1^{\sigma}(t)\, g(t)}{W^{\sigma}(y_1, y_2)(t)}.$$

Integrating directly yields the following results.

Theorem 2.16. (Variation of parameters method) [4] Let $t_0 \in \mathbb{T}^k$. Assume that y_1 and y_2 form a fundamental set of solutions of the homogeneous equation $L_2 y(t) = 0$. Then, the solution of the IVP

$$L_2 y(t) = g(t),\ y(t_0) = y_0,\ y^{\Delta}(t_0) = y_0^{\Delta},$$

where $y_0, y_0^{\Delta} \in \mathbb{R}$ is given by

$$y(t) = \alpha_0 y_1(t) + \beta_0 y_2(t) + \int_{t_0}^{t} \frac{y_1(\sigma(\tau)) y_2(t) - y_2(\sigma(\tau)) y_1(t)}{W(y_1, y_2)(\sigma(\tau))} g(\tau) \Delta\tau,$$

where the constants α_0 and β_0 are

$$\begin{aligned} \alpha_0 &= \frac{y_2^{\Delta}(t_0) y_0 - y_2(t_0) y_0^{\Delta}}{W(y_1, y_2)(t_0)}, \\ \beta_0 &= \frac{y_1(t_0)(t) y_0^{\Delta} - y_1^{\Delta}(t_0) y_0}{W(y_1, y_2)(t_0)}. \end{aligned}$$

Example 2.11. [4] Solve

$$y^{\Delta\Delta} - 5y^{\Delta} + 6y = e_4(t, t_0). \tag{2.8}$$

by using variation of parameters method.

Solution: The general solution of the corresponding nonhomogeneous equation is

$$y(t) = c_1 e_2(t, t_0) + c_2 e_3(t, t_0).$$

Hence, there is a solution of the nonhomogeneous equation (2.8) of the form

$$y_p(t) = \alpha(t)\, e_2(t, t_0) + \beta(t)\, e_3(t, t_0),$$

where $\alpha(t)$ and $\beta(t)$ are chosen to satisfy the system of equations

$$\alpha^{\Delta}(t)\, e_2^{\sigma}(t, t_0) + \beta^{\Delta}(t)\, e_3^{\sigma}(t, t_0) = 0, \tag{2.9}$$

$$2\alpha^{\Delta}(t)\, e_2^{\sigma}(t, t_0) + 3\beta^{\Delta}(t)\, e_3^{\sigma}(t, t_0) = e_4(t, t_0). \tag{2.10}$$

Multiplying equation (2.9) by 3 and substracting from equation (2.10) leads to

$$\alpha^{\Delta}(t) = -\frac{e_4(t,t_0)}{e_2^{\sigma}(t,t_0)}.$$

Integrating and applying remark (2.9), we see that we can take

$$\alpha(t) = -\frac{1}{2} e_{\frac{2}{1+2\mu(t)}}(t,t_0).$$

Multiplying equation (2.9) by 2 and substracting from equation (2.10) leads to

$$\beta^{\Delta}(t) = \frac{e_4(t,t_0)}{e_3^{\sigma}(t,t_0)}.$$

Integrating and applying remark (2.9), we see that we can take

$$\beta(t) = e_{\frac{1}{1+3\mu(t)}}(t,t_0).$$

It follows that

$$\begin{aligned} y_p(t) &= -\frac{1}{2} e_{\frac{2}{1+2\mu(t)}}(t,t_0) e_2(t,t_0) + e_{\frac{1}{1+3\mu(t)}}(t,t_0) e_3(t,t_0) \\ &= \frac{1}{2} e_4(t,t_0). \end{aligned}$$

Hence, the general solution of the given equation is given by

$$y(t) = c_1 e_2(t,t_0) + c_2 e_3(t,t_0) + \frac{1}{2} e_4(t,t_0).$$

Remark 2.9. [4] If $p, q \in \mathcal{R}$, then

$$e^{\Delta}_{p \ominus q}(.,t_0) = (p-q)\frac{e_p(t,t_0)}{e_q^{\sigma}(t,t_0)}.$$

3 FIRST AND SECOND CANONIC FORMS OF DIRAC SYSTEM ON TIME SCALES

In this chapter, we try to prove some fundamental spectral properties of Dirac eigenvalue problem of the form

$$Ly = By^{\Delta}(t) + Q(t)y^{\sigma}(t) = \lambda y^{\sigma}(t), t \in [\rho(a), b] \cap \mathbb{T} \tag{3.1}$$

with the separated boundary conditions

$$\alpha y_1(\rho(a)) + \beta y_2(\rho(a)) = 0, \tag{3.2}$$

$$\gamma y_1(b) + \delta y_2(b) = 0, \tag{3.3}$$

where

$$Q(t) = \begin{pmatrix} q(t) & 0 \\ 0 & r(t) \end{pmatrix}, \ B = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix},$$

and λ is a spectral parameter. Throughout this study, we will assume that $q, r : [\rho(a), b] \cap \mathbb{T} \to \mathbb{R}$ are continuous functions; $a, b \in \mathbb{T}$ with $a < b$, $y^{\sigma} = y(\sigma)$ and $\left(\alpha^2 + \beta^2\right)\left(\gamma^2 + \delta^2\right) \neq 0$. $y(t) = \begin{pmatrix} y_1(t), & y_2(t) \end{pmatrix}^T$ is known as eigenfunction of the problem (3.1)-(3.3), where T denotes transpose. After some straightforward computations in (3.1), one can easily get the following system

$$\begin{aligned} y_2^{\Delta} &= (\lambda - q(t))\, y_1^{\sigma}, \\ y_1^{\Delta} &= (-\lambda + r(t))\, y_2^{\sigma}. \end{aligned} \tag{3.4}$$

By taking $\mathbb{T} = \mathbb{R}$ in (3.4), we obtain below classical Dirac system

$$\begin{aligned} y_2' &= (\lambda - q(t))\, y_1, \\ y_1' &= (-\lambda + r(t))\, y_2. \end{aligned} \tag{3.5}$$

The system (3.5) is known as the first canonic form of the Dirac system in literature. There are several studies about this system in spectral theory. Now, we will give some information related to the historical development and physical meaning of Dirac system in classical spectral theory.

Dirac operator is the relativistic Schrödinger operator in classical quantum physics. It is a modern impression of the relativistic quantum mechanics of electrons intended to make original mathematical results [23]. It treats in some depth the relativistic invariance of a quantum theory, self-adjointness and spectral theory, qualitative features of relativistic bound and scattering states, and the external field problem in quantum electrodynamics, without neglecting the interpretational difficulties and restrictions of the theory. Particularly, some aspects of inverse eigenvalue problems for Dirac system had been considered by Moses [24], Prats and Toll [25], Verde [26], Gasymov, Levitan [27] and Panakhov [28]. It is well known that two spectra uniquely determine the matrix valued potential function [29]. The fundamental and exhaustive results about Dirac operators were given in [30]. In [31], eigenfunction expansion for one dimensional Dirac operators describing the motion of a particle in quantum mechanics is obtained. Inverse spectral problems for Dirac system without spectral parameter in boundary condition have been considered by various authors (see [32], [33], [34], [35], [36], [37], [38]).

Problems with a spectral parameter in equations and boundary conditions form an important part of spectral theory of linear differential operators. Inverse eigenvalue problem for Dirac system with a spectral parameter in the boundary conditions was considered by Kerimov [39]. Inverse nodal problem for Dirac system with a spectral parameter in the boundary conditions was considered in [40], where it was proved that a set of nodal points of one of the components of the eigenfunctions uniquely determines all the parameters of the boundary conditions and the coefficients of the Dirac system. Inverse nodal problem for the Dirac system with boundary conditions depending polynomially on the spectral parameter was considered by Guo and Wei [41].

This chapter is arranged as follows: In Section 3.1, we prove some basic theorems for the first canonic form of the Dirac system on $\mathbb{T}$. Using some methods, we get asymptotic estimates of eigenfunction for the first and second canonic forms of the eigenvalue problem (3.1)-(3.3) in Section 3.2. In Section 3.3, we give some fundamental properties of second canonic form of Dirac system on time scales.

3.1 Some spectral properties for the first canonic form of Dirac system on time scales

Here, we give some valuable results for the first canonic form of Dirac system on a time scale $\mathbb{T}$ whose all points are right dense. It is well known that the problem (3.1)-(3.3) has only real, simple eigenvalues and its eigenfunctions are orthogonal in case of $\mathbb{T} = \mathbb{R}$ [40]. The below results generalize this classical basic results.

Theorem 3.1. **[20]** The eigenvalues of the problem (3.1)-(3.3) are all simple.

Proof: Let $\lambda, \mu \in \mathbb{R}$ be spectral parameters where $\lambda \neq \mu$ and $u(t) = \begin{pmatrix} u_1(t), & u_2(t) \end{pmatrix}^T$ be eigenfunction of (3.1)-(3.3). Then, we have

$$\{u_1(t,\lambda)u_2^\sigma(t,\mu) - u_1(t,\mu)u_2^\sigma(t,\lambda)\}^\Delta = (\mu - \lambda)\left(u_1^\sigma(t,\lambda)u_1^\sigma(t,\mu) + u_2^\sigma(t,\lambda)u_2^\sigma(t,\mu)\right). \qquad (3.1.1)$$

By taking $\Delta-$integral of (3.1.1) on $[\rho(a), b]$, we get

$$\begin{aligned}(\mu-\lambda)\int\limits_{\rho(a)}^{b}\left(u_1^\sigma(t,\lambda)u_1^\sigma(t,\mu)+u_2^\sigma(t,\lambda)u_2^\sigma(t,\mu)\right)\Delta t &= u_1(b,\lambda)u_2^\sigma(b,\mu)-u_1(b,\mu)u_2^\sigma(b,\lambda)\\ &\quad -u_1(\rho(a),\lambda)u_2^\sigma(\rho(a),\mu)+u_1(\rho(a),\mu)u_2^\sigma(\rho(a),\lambda).\end{aligned}$$

Then, as $\mu\to\lambda$, it yields

$$-\int\limits_{\rho(a)}^{b}\left\{|u_1^\sigma(t,\lambda)|^2+|u_2^\sigma(t,\lambda)|^2\right\}\Delta t = u_1(b,\lambda)\frac{\partial}{\partial\lambda}u_2^\sigma(b,\lambda)-u_2^\sigma(b,\lambda)\frac{\partial}{\partial\lambda}u_1^\sigma(b,\lambda). \tag{3.1.2}$$

Therefore, $\Lambda(\lambda)=-\left[\gamma u_1(b,\lambda)+\delta u_2(b,\lambda)\right]$ has only simple zeros. Indeed, conversely, suppose that λ^* be double-decker root. Then,

$$\left.\begin{array}{r}\gamma u_1(b,\lambda^*)+\delta u_2(b,\lambda^*)=0\\ \gamma\dfrac{\partial}{\partial\lambda}u_1(b,\lambda^*)+\delta\dfrac{\partial}{\partial\lambda}u_2(b,\lambda^*)=0\end{array}\right\}\Rightarrow u_2(b,\lambda^*)\frac{\partial}{\partial\lambda}u_1(b,\lambda^*)-u_1(b,\lambda^*)\frac{\partial}{\partial\lambda}u_2(b,\lambda^*)=0. \tag{3.1.3}$$

By (3.1.2) and (3.1.3), we obtain

$$\int\limits_{\rho(a)}^{b}\left\{|u_1^\sigma(t,\lambda)|^2+|u_2^\sigma(t,\lambda)|^2\right\}\Delta t=0\Rightarrow u_1^\sigma(t,\lambda)=u_2^\sigma(t,\lambda)=0,$$

for $\lambda=\lambda^*$. This is a contradiction. So, $\Lambda(\lambda)=\gamma u_1(b)+\delta u_2(b)$ has only simple zeros.

Theorem 3.2. **[20]** All eigenvalues of the problem (3.1)-(3.3) are real.

Proof: Let $\overline{\lambda}_0$ be a complex eigenvalue and $\overline{u}(t)=\begin{pmatrix}\overline{u}_1(t), & \overline{u}_2(t)\end{pmatrix}^T$ be an eigenfunction corresponding to the eigenvalue $\overline{\lambda}_0$ of the problem (3.1)-(3.3). Then, we obtain

$$\{u_1(t)\overline{u}_2^\sigma(t)-\overline{u}_1(t)u_2^\sigma(t)\}^\Delta=u_1^\Delta\overline{u}_2^\sigma+u_1^\sigma\overline{u}_2^{\sigma\Delta}-\overline{u}_1^\Delta u_2^\sigma-\overline{u}_1^\sigma u_2^{\sigma\Delta}=\left(\overline{\lambda}_0-\lambda_0\right)\left(|u_1^\sigma(t)|^2+|u_2^\sigma(t)|^2\right).$$

If we take $\Delta-$integral of the last equality from $\rho(a)$ to b, we get

$$\begin{aligned}\left(\overline{\lambda}_0-\lambda_0\right)\int\limits_{\rho(a)}^{b}\left(|u_1^\sigma(t)|^2+|u_2^\sigma(t)|^2\right)\Delta t &= u_1(b)\overline{u}_2^\sigma(b)-\overline{u}_1(b)u_2^\sigma(b)\\ &\quad -u_1(\rho(a))\overline{u}_2^\sigma(\rho(a))+\overline{u}_1(\rho(a))u_2^\sigma(\rho(a))\\ &= 0,\end{aligned}$$

by considering the boundary conditions (3.2), (3.3). So, we have

$$\int\limits_{\rho(a)}^{b}\left(|u_1^\sigma(t)|^2+|u_2^\sigma(t)|^2\right)\Delta t=0,$$

and

$$u_1^\sigma(t)=0 \text{ and } u_2^\sigma(t)=0,$$

for $\overline{\lambda}_0 \neq \lambda_0$. This is a contradiction. Hence, all eigenvalues of the problem (3.1)-(3.3) are real.

Theorem 3.3. [20] Let $u=\begin{pmatrix} u_1, & u_2 \end{pmatrix}^T, v=\begin{pmatrix} v_1, & v_2 \end{pmatrix}^T \in C_{rd}^1(\mathbb{T})$ be the eigenfunctions of the problem (3.1)-(3.3). Then,

a) $(Lu)^T v^\sigma - (Lv)^T u^\sigma = W^\Delta(u,v)$ on $[\rho(a),b]\cap\mathbb{T}$.

b) $< Lu, v^\sigma > - < Lv, u^\sigma >= W(u,v)(b) - W(u,v)(\rho(a))$,

where $W(u,v)=u_2v_1^\sigma - u_1v_2^\sigma$.

Proof:

a) Definition of W and product rule for $\Delta-$derivative give

$$\begin{aligned} W^\Delta(u,v) &= -\left\{v_2^\Delta u_1^\sigma - v_1^\Delta u_2^\sigma - u_2^\Delta v_1^\sigma + u_1^\Delta v_2^\sigma\right\} \\ &= -u_1^\sigma\left(v_2^\Delta + qv_1^\sigma\right) - u_2^\sigma\left(-v_1^\Delta + rv_2^\sigma\right) + v_1^\sigma\left(u_2^\Delta + qu_1^\sigma\right) + v_2^\sigma\left(-u_1^\Delta + ru_2^\sigma\right) \\ &= (Lu)^T v^\sigma - (Lv)^T u^\sigma. \end{aligned}$$

b) By using definition of W and inner product on the set of so-called $rd-$continuous functions, we have

$$< Lu, v^\sigma > - < Lv, u^\sigma >= \int_{\rho(a)}^{b} \left[(Lu)^T v^\sigma - (Lv)^T u^\sigma\right]\Delta t$$

$$\begin{aligned} &= \int_{\rho(a)}^{b} \left\{\left(Bu^\Delta + Qu^\sigma\right)^T v^\sigma - \left(Bv^\Delta + Qv^\sigma\right)^T u^\sigma\right\}\Delta t \\ &= -\int_{\rho(a)}^{b} \left\{v_2^\Delta u_1^\sigma - v_1^\Delta u_2^\sigma - u_2^\Delta v_1^\sigma + u_1^\Delta v_2^\sigma\right\}\Delta t \\ &= \int_{\rho(a)}^{b} \left\{W(u,v)\right\}^\Delta \Delta t \\ &= W(u,v)(b) - W(u,v)(\rho(a)). \end{aligned}$$

Hence, this completes the proof. In classical spectral theory, these equalities are known as Lagrange's identity and Green's formula, respectively.

Theorem 3.4. [20] The equality

$$u_2(t,\lambda)\frac{\partial}{\partial\lambda}u_1(t,\lambda) - u_1(t,\lambda)\frac{\partial}{\partial\lambda}u_2(t,\lambda) = -\int_{\rho(a)}^{t} \left\{[u_1^\sigma(\tau,\lambda)]^2 + [u_2^\sigma(\tau,\lambda)]^2\right\}\Delta\tau,$$

holds for all $t \in [\rho(a), b] \cap \mathbb{T}$ and $\lambda \in \mathbb{R}$.

Proof: Let v, $\lambda \in \mathbb{R}$ with $v \neq \lambda$. Then,

$$\{u_1(t,\lambda)\, u_2(t,v) - u_1(t,v)\, u_2(t,\lambda)\}^{\Delta} = (v-\lambda)\left\{u_2^{\sigma}(t,\lambda)\, u_2^{\sigma}(t,v) + u_1^{\sigma}(t,\lambda)\, u_1^{\sigma}(t,v)\right\}.$$

Dividing both sides of above equality by $\lambda - v$ and taking limit as $v \to \lambda$, we have

$$\lim_{v\to\lambda} \frac{\{u_1(t,\lambda)\, u_2(t,v) - u_1(t,v)\, u_2(t,\lambda)\}^{\Delta}}{\lambda - v} = -\lim_{v\to\lambda} \left\{u_1^{\sigma}(t,\lambda)\, u_1^{\sigma}(t,v) + u_2^{\sigma}(t,\lambda)\, u_2^{\sigma}(t,v)\right\}$$

$$\Rightarrow \left\{u_2(t,\lambda)\frac{\partial}{\partial\lambda}u_1(t,\lambda) - u_1(t,\lambda)\frac{\partial}{\partial\lambda}u_2(t,\lambda)\right\}^{\Delta} = -\left\{[u_1^{\sigma}(t,\lambda)]^2 + [u_2^{\sigma}(t,\lambda)]^2\right\}.$$

By taking $\Delta-$integral of the last equality from $\rho(a)$ to t, we get

$$\int_{\rho(a)}^{t} \left\{u_2(\tau,\lambda)\frac{\partial}{\partial\lambda}u_1(\tau,\lambda) - u_1(\tau,\lambda)\frac{\partial}{\partial\lambda}u_2(\tau,\lambda)\right\}^{\Delta} \Delta\tau = -\int_{\rho(a)}^{t} \left\{[u_1^{\sigma}(\tau,\lambda)]^2 + [u_2^{\sigma}(\tau,\lambda)]^2\right\} \Delta\tau.$$

Since $u_1(\rho(a),\lambda) = \beta$ and $u_2(\rho(a),\lambda) = -\alpha$, it yields $\frac{\partial}{\partial\lambda}u_1(\rho(a),\lambda) = 0$ and $\frac{\partial}{\partial\lambda}u_2(\rho(a),\lambda) = 0$. Finally, after some computations, we obtain

$$u_2(t,\lambda)\frac{\partial}{\partial\lambda}u_1(t,\lambda) - u_1(t,\lambda)\frac{\partial}{\partial\lambda}u_2(t,\lambda) = -\int_{\rho(a)}^{t} \left\{[u_1^{\sigma}(\tau,\lambda)]^2 + [u_2^{\sigma}(\tau,\lambda)]^2\right\} \Delta\tau.$$

So, the proof is complete.

Theorem 3.5. **[20]** The eigenfunctions

$$y(t,\lambda_1) = \begin{pmatrix} y_1(t,\lambda_1), & y_2(t,\lambda_1) \end{pmatrix}^T \text{ and } z(t,\lambda_2) = \begin{pmatrix} z_1(t,\lambda_2), & z_2(t,\lambda_2) \end{pmatrix}^T,$$

of the problem (3.1)-(3.3) corresponding to distinct eigenvalues λ_1 and λ_2, are orthogonal , i.e

$$\int_{\rho(a)}^{b} y^T(t,\lambda_1) z(t,\lambda_2)\Delta t = 0.$$

Proof: Let us use following equality

$$\{y_1(t,\lambda_1)z_2(t,\lambda_2) - y_2(t,\lambda_1)z_1(t,\lambda_2)\}^{\Delta} = (\lambda_2 - \lambda_1)\left[y_1^{\sigma}(t,\lambda_1)z_1(t,\lambda_2) + y_2^{\sigma}(t,\lambda_1)z_2(t,\lambda_2)\right].$$

Taking $\Delta-$integral of the last equality from $\rho(a)$ to b, we get

$$\begin{aligned} \int_{\rho(a)}^{b} \{y_1(t,\lambda_1)z_1(t,\lambda_2) + y_2(t,\lambda_1)z_2(t,\lambda_2)\}\Delta t &= \int_{\rho(a)}^{b} y^T(t,\lambda_1)z(t,\lambda_2)\Delta t \\ &= 0, \end{aligned}$$

for $\lambda_1 \neq \lambda_2$. Then, it shows that the eigenfunctions $y(t,\lambda_1)$ and $z(t,\lambda_2)$ corresponding to distinct eigenvalues are always orthogonal.

3.2 Main Results for the first canonic form of Dirac system on time scales

In this section, we get asymptotic estimates for the component eigenfunctions $y_1(t,t_0)$ and $y_2(t,t_0)$ of the problem (3.1)-(3.3) on $\mathbb{T}$ where $y(t,t_0) = \begin{pmatrix} y_1(t,t_0), & y_2(t,t_0) \end{pmatrix}^T$.

Theorem 3.6. [20] The component eigenfunctions $y_1(t,t_0)$ and $y_2(t,t_0)$ of the problem (3.1)-(3.3) have the following asymptotic estimates;

$$y_1(t,t_0) = -\alpha \sin_\lambda(t,\rho(a)) + \beta \cos_\lambda(t,\rho(a)) + \int_{\rho(a)}^{t} \left[-q(s)y_1(s,t_0)\sin_\lambda(t,s) + r(s)y_2(s,t_0)\cos_\lambda(t,s)\right]\Delta s,$$

and

$$y_2(t,t_0) = -\alpha \cos_\lambda(t,\rho(a)) - \beta \sin_\lambda(t,\rho(a)) + \int_{\rho(a)}^{t} \left[-q(s)y_1(s,t_0)\cos_\lambda(t,s) - r(s)y_2(s,t_0)\sin_\lambda(t,s)\right]\Delta s,$$

where $t, t_0 \in [\rho(a), b] \cap \mathbb{T}$.

Proof: Let us consider the system (3.4). Firstly, we need to find the homogeneous solution of this system. By homogeneous form of (3.4), one can easily get

$$y_2^{\Delta\Delta} + \lambda^2 y_2 = 0.$$

The homogeneous solution of above equation is

$$y_2(t,t_0) = c_1 \cos_\lambda(t,t_0) + c_2 \sin_\lambda(t,t_0). \tag{3.2.1}$$

Since $y_2^\Delta = \lambda y_1$, we have

$$y_1(t,t_0) = -c_1 \sin_\lambda(t,t_0) + c_2 \cos_\lambda(t,t_0), \tag{3.2.2}$$

where c_1 and c_2 are arbitrary constants. Now, we are ready to apply variation of parameters method (see [9]) to find particular solutions of the system (3.4). By (3.4), we can have

$$\begin{aligned} -c_1^\Delta \sin_\lambda(t,t_0) + c_2^\Delta \cos_\lambda(t,t_0) &= -q(t)y_1, \\ c_1^\Delta \cos_\lambda(t,t_0) + c_2^\Delta \sin_\lambda(t,t_0) &= r(t)y_2. \end{aligned}$$

By multiplying these equations by $-\sin_\lambda(t,t_0)$, $\cos_\lambda(t,t_0)$ and $\cos_\lambda(t,t_0), \sin_\lambda(t,t_0)$, respectively, we get

$$\begin{aligned} c_1^\Delta &= q(t)y_1 \sin_\lambda(t,t_0) + r(t)y_2 \cos_\lambda(t,t_0), \\ c_2^\Delta &= -q(t)y_1 \cos_\lambda(t,t_0) + r(t)y_2 \sin_\lambda(t,t_0). \end{aligned}$$

Integrating these equalities and considering (3.2.1) and (3.2.2), we obtain

$$\begin{aligned}
y_1(t,t_0) &= -\sin_\lambda(t,t_0)\int\limits_{\rho(a)}^{t}\left[q(s)y_1(s,t_0)\sin_\lambda(s,t_0)+r(s)y_2(s,t_0)\cos_\lambda(s,t_0)\right]\Delta s \\
&\quad +\cos_\lambda(t,t_0)\int\limits_{\rho(a)}^{t}\left[-q(s)y_1(s,t_0)\cos_\lambda(s,t_0)+r(s)y_2(s,t_0)\sin_\lambda(s,t_0)\right]\Delta s \\
&= -\int\limits_{\rho(a)}^{t}\left\{q(s)y_1(s,t_0)\cos_\lambda(t,s)+r(s)y_2(s,t_0)\sin_\lambda(t,s)\right\}\Delta s, \qquad (3.2.3)
\end{aligned}$$

and

$$\begin{aligned}
y_2(t,t_0) &= \cos_\lambda(t,t_0)\int\limits_{\rho(a)}^{t}\left[q(s)y_1(s,t_0)\sin_\lambda(s,t_0)+r(s)y_2(s,t_0)\cos_\lambda(s,t_0)\right]\Delta s \qquad (3.2.4) \\
&\quad +\sin_\lambda(t,t_0)\int\limits_{\rho(a)}^{t}\left[-q(s)y_1(s,t_0)\cos_\lambda(s,t_0)+r(s)y_2(s,t_0)\sin_\lambda(s,t_0)\right]\Delta s \\
&= \int\limits_{\rho(a)}^{t}\left\{-q(s)y_1(s,t_0)\sin_\lambda(t,s)+r(s)y_2(s,t_0)\cos_\lambda(t,s)\right\}\Delta s.
\end{aligned}$$

After using integration by parts in (3.2.3) and (3.2.4), respectively, we get

$$y_1 = y_2(t,t_0)+\alpha\cos_\lambda(t,\rho(a))+\beta\sin_\lambda(t,\rho(a)), \qquad (3.2.5)$$

and

$$y_2 = y_1(t,t_0)+\alpha\sin_\lambda(t,\rho(a))-\beta\cos_\lambda(t,\rho(a)). \qquad (3.2.6)$$

Therefore, by considering (3.2.4), (3.2.6) and (3.2.3), (3.2.5), we have

$$y_1(t,t_0) = -\alpha\sin_\lambda(t,\rho(a))+\beta\cos_\lambda(t,\rho(a))+\int\limits_{\rho(a)}^{t}\left[-q(s)y_1(s,t_0)\sin_\lambda(t,s)+r(s)y_2(s,t_0)\cos_\lambda(t,s)\right]\Delta s,$$

and

$$y_2(t,t_0) = -\alpha\cos_\lambda(t,\rho(a))-\beta\sin_\lambda(t,\rho(a))+\int\limits_{\rho(a)}^{t}\left[-q(s)y_1(s,t_0)\cos_\lambda(t,s)-r(s)y_2(s,t_0)\sin_\lambda(t,s)\right]\Delta s.$$

This completes proof.

3.3 Some Basic results for the second canonic form of Dirac system on time scales

Here, let us define second canonic form of Dirac eigenvalue problem on a time scale as following

$$Ly = By^{\Delta}(t) + Q(t)y^{\sigma}(t) = \lambda y^{\sigma}(t),\ t \in [\rho(a), b] \cap \mathbb{T} \tag{3.3.1}$$

with the separated boundary conditions

$$\alpha y_1(\rho(a)) + \beta y_2(\rho(a)) = 0, \tag{3.3.2}$$

$$\gamma y_1(b) + \delta y_2(b) = 0, \tag{3.3.3}$$

where

$$Q(t) = \begin{pmatrix} p(t) & q(t) \\ q(t) & -p(t) \end{pmatrix}, B = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix},$$

and $p, q : \mathbb{T} \to \mathbb{R}$ are continuous functions. $\alpha, \beta, \gamma, \delta$ are constants with the same properties in (3.1)-(3.3). If we make some computations in (3.3.1), we get the following system

$$\begin{aligned} y_2^{\Delta} + p(t)y_1^{\sigma} + q(t)y_2^{\sigma} &= \lambda y_1^{\sigma} \\ -y_1^{\Delta} + q(t)y_1^{\sigma} - p(t)y_2^{\sigma} &= \lambda y_2^{\sigma}. \end{aligned} \tag{3.3.4}$$

The system (3.3.4) is called second canonic form of Dirac system for $\mathbb{T} = \mathbb{R}$ in spectral theory. Similarly, following theorems can be easily proved for second canonic form of Dirac system on $\mathbb{T}$.

Theorem 3.7. **[20]** All eigenvalues of the problem (3.3.1)-(3.3.3) are simple.

Proof: Let $\lambda, \mu \in \mathbb{R}$ be spectral parameters where $\lambda \neq \mu$ and $\varphi(t) = \begin{pmatrix} \varphi_1(t), & \varphi_2(t) \end{pmatrix}^T$ be eigenfunction of (3.3.1)-(3.3.3). Then, we have

$$\frac{d}{dx}\left\{\varphi_1(x,\lambda)\varphi_2^{\sigma}(x,\mu) - \varphi_1(x,\mu)\varphi_2^{\sigma}(x,\lambda)\right\} = (\mu - \lambda)\left(\varphi_1^{\sigma}(x,\lambda)\varphi_1^{\sigma}(x,\mu) + \varphi_2^{\sigma}(x,\lambda)\varphi_2^{\sigma}(x,\mu)\right). \tag{3.3.5}$$

By taking $\Delta-$integral of (3.3.5) on $[\rho(a), b]$, we get

$$\begin{aligned} (\mu - \lambda)\int\limits_{\rho(a)}^{b}\left(\varphi_1^{\sigma}(x,\lambda)\varphi_1^{\sigma}(x,\mu) + \varphi_2^{\sigma}(x,\lambda)\varphi_2^{\sigma}(x,\mu)\right)\Delta x &= \varphi_1(b,\lambda)\varphi_2^{\sigma}(b,\mu) - \varphi_1(b,\mu)\varphi_2^{\sigma}(b,\lambda) \\ &\quad -\varphi_1(\rho(a),\lambda)\varphi_2^{\sigma}(\rho(a),\mu) \\ &\quad +\varphi_1(\rho(a),\mu)\varphi_2^{\sigma}(\rho(a),\lambda). \end{aligned}$$

Then, as $\mu \to \lambda$, it yields

$$\int_{\rho(a)}^{b} \left(|\varphi_1^\sigma(x,\lambda)|^2 + |\varphi_2^\sigma(x,\lambda)|^2\right)\Delta x = \varphi_1(b,\lambda)\frac{\partial}{\partial\lambda}\varphi_2^\sigma(b,\lambda) - \varphi_2^\sigma(b,\lambda)\frac{\partial}{\partial\lambda}\varphi_1^\sigma(b,\lambda). \tag{3.3.6}$$

Therefore, $\Lambda(\lambda) = -[\gamma\varphi_1(b,\lambda) + \delta\varphi_2(b,\lambda)]$ has only simple zeros. Indeed, conversely, suppose that λ^* be double-decker root. Then,

$$\left.\begin{array}{c} \gamma\varphi_1(b,\lambda^*) + \delta\varphi_2(b,\lambda^*) \\ \gamma\frac{\partial}{\partial\lambda}\varphi_1(b,\lambda^*) + \delta\frac{\partial}{\partial\lambda}\varphi_2(b,\lambda^*) = 0 \end{array}\right\} \Rightarrow \varphi_2(b,\lambda^*)\frac{\partial}{\partial\lambda}\varphi_1(b,\lambda^*) - \varphi_1(b,\lambda^*)\frac{\partial}{\partial\lambda}\varphi_2(b,\lambda^*) = 0 \tag{3.3.7}$$

From (3.3.6) and (3.3.7), we get that

$$\int_{\rho(a)}^{b} \left(|\varphi_1^\sigma(x,\lambda)|^2 + |\varphi_2^\sigma(x,\lambda)|^2\right)\Delta x = 0 \Longrightarrow \varphi_1^\sigma(x,\lambda) = \varphi_2^\sigma(x,\lambda) = 0,$$

for $\lambda = \lambda^*$. This is a contradiction. So, $\Lambda(\lambda) = \gamma u_1(b) + \delta u_2(b)$ has only simple zeros.

Theorem 3.8. **[20]** The eigenvalues of the problem (3.3.1)-(3.3.3) are real.

Proof: Let $\bar{\lambda}_0$ be a complex eigenvalue and $\overline{u}(t) = \begin{pmatrix} \overline{u}_1(t), & \overline{u}_2(t) \end{pmatrix}^T$ be an eigenfunction corresponding to the eigenvalue $\bar{\lambda}_0$ of the problem (3.3.1)-(3.3.3). Then we obtain

$$\begin{aligned} \frac{d}{dt}\{u_1(t)\overline{u}_2^\sigma(t) - \overline{u}_1(t)u_2^\sigma(t)\} &= u_1^\Delta\overline{u}_2^\sigma + u_1^\sigma\overline{u}_2^{\sigma\Delta} - \overline{u}_1^\Delta u_2^\sigma - \overline{u}_1^\sigma u_2^{\sigma\Delta} \\ &= [q(x)u_1^\sigma - p(x)u_2^\sigma - \lambda_0 u_2^\sigma]\,\overline{u}_2^\sigma + \left[-p(x)\overline{u}_1^\sigma - q(x)\overline{u}_2^\sigma + \bar{\lambda}_0\overline{u}_1^\sigma\right]u_1^\sigma \\ &\quad - \left[q(x)\overline{u}_1^\sigma - p(x)\overline{u}_2^\sigma - \bar{\lambda}_0\overline{u}_2^\sigma\right]u_2^\sigma - [-p(x)u_1^\sigma - q(x)u_2^\sigma + \lambda_0 u_1^\sigma]\,\overline{u}_1^\sigma \\ &= \left(\bar{\lambda}_0 - \lambda_0\right)(u_1^\sigma\overline{u}_1^\sigma + u_2^\sigma\overline{u}_2^\sigma) = \left(\bar{\lambda}_0 - \lambda_0\right)\left(|u_1^\sigma|^2 + |u_2^\sigma|^2\right) \end{aligned}$$

and

$$\begin{aligned} \left(\bar{\lambda}_0 - \lambda_0\right)\int_{\rho(a)}^{b}\left(|u_1^\sigma(t)|^2 + |u_2^\sigma(t)|^2\right)\Delta t &= u_1(b)\,\overline{u}_2^\sigma(b) - \bar{u}_1(b)\,u_2^\sigma(b) \\ &\quad - u_1(\rho(a))\,\overline{u}_2^\sigma(\rho(a)) + \bar{u}_1(\rho(a))\,u_2^\sigma(\rho(a)) \\ &= \delta(-\gamma) - \delta(-\gamma) - \beta(-\alpha) + \beta(-\alpha) = 0. \end{aligned}$$

by considering the boundary conditions (3.3.2), (3.3.3). Since $\bar{\lambda}_0 \neq \lambda_0$, we have

$$\int_{\rho(a)}^{b}\left(|u_1^\sigma(t)|^2 + |u_2^\sigma(t)|^2\right)\Delta t = 0 \Longrightarrow |u_1^\sigma(t)|^2 + |u_2^\sigma(t)|^2 = 0 \Longrightarrow u_1^\sigma(t) = u_2^\sigma(t) = 0.$$

This completes the proof.

Theorem 3.9. **[20]** The eigenfunctions

$$y(t,\lambda_1) = \begin{pmatrix} y_1(t,\lambda_1), & y_2(t,\lambda_1) \end{pmatrix}^T \text{ and } z(t,\lambda_2) = \begin{pmatrix} z_1(t,\lambda_2), & z_2(t,\lambda_2) \end{pmatrix}^T,$$

of the problem (3.3.1)-(3.3.3) corresponding to distinct eigenvalues λ_1 and λ_2 are orthogonal, i.e

$$\int_{\rho(a)}^{b} y^T(t,\lambda_1) z(t,\lambda_2)\Delta t = 0.$$

Theorem 3.10. **[20]** Let $u = \begin{pmatrix} u_1, & u_2 \end{pmatrix}^T, v = \begin{pmatrix} v_1, & v_2 \end{pmatrix}^T \in C^1_{rd}(\mathbb{T})$ be the eigenfunctions of the problem (3.3.1)-(3.3.3). Then,

a) $(Lu)^T v^\sigma - (Lv)^T u^\sigma = W^\Delta(u,v)$ on $[\rho(a), b] \cap \mathbb{T}$.

b) $< Lu, v^\sigma > - < Lv, u^\sigma > = W(u,v)(b) - W(u,v)(\rho(a))$,

where $W(u,v) = u_2 v_1^\sigma - u_1 v_2^\sigma$.

Proof: a) Definition of W and product rule for $\Delta-$derivative give

$$\begin{aligned} W^\Delta(u,v) &= -\left\{ v_2^\Delta u_1^\sigma - v_1^\Delta u_2^\sigma - u_2^\Delta v_1^\sigma + u_1^\Delta v_2^\sigma \right\} \\ &= -u_1^\sigma \left(v_2^\Delta + p v_1^\sigma + q v_2^\sigma \right) - u_2^\sigma \left(-v_1^\Delta + q v_1^\sigma - p v_2^\sigma \right) \\ &\quad + v_1^\sigma \left(u_2^\Delta + p u_1^\sigma + q u_2^\sigma \right) + v_2^\sigma \left(-u_1^\Delta + q u_1^\sigma - p u_2^\sigma \right) \\ &= (Lu)^T v^\sigma - (Lv)^T u^\sigma. \end{aligned}$$

b) By using the definition of Wronskian and inner product, we have

$$\begin{aligned} \langle Lu, v^\sigma \rangle - \langle Lv, u^\sigma \rangle &= \int_{\rho(a)}^{b} \left[(Lu)^T v^\sigma - (Lv)^T u^\sigma \right] \Delta t \\ &= \int_{\rho(a)}^{b} \left\{ \left[Bu^\Delta + Q(x) u^\sigma \right]^T v^\sigma - \left[Bv^\Delta + Q(x) v^\sigma \right]^T u^\sigma \right\} \Delta t \end{aligned}$$

$$\begin{aligned} = &\int_{\rho(a)}^{b} \left[\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \begin{pmatrix} u_1^\Delta \\ u_2^\Delta \end{pmatrix} + \begin{pmatrix} p(t) & q(t) \\ q(t) & -p(t) \end{pmatrix} \begin{pmatrix} u_1^\sigma \\ u_2^\sigma \end{pmatrix} \right]^T \begin{pmatrix} v_1^\sigma \\ v_2^\sigma \end{pmatrix} \Delta t \\ &- \int_{\rho(a)}^{b} \left[\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \begin{pmatrix} v_1^\Delta \\ v_2^\Delta \end{pmatrix} + \begin{pmatrix} p(t) & q(t) \\ q(t) & -p(t) \end{pmatrix} \begin{pmatrix} v_1^\sigma \\ v_2^\sigma \end{pmatrix} \right]^T \begin{pmatrix} u_1^\sigma \\ u_2^\sigma \end{pmatrix} \Delta t \end{aligned}$$

$$= \int_{\rho(a)}^{b} \left[\left(u_2^{\Delta}+p(t)u_1^{\sigma}+q(t)u_2^{\sigma}\right)v_1^{\sigma}+\left(-u_1^{\Delta}+q(t)u_1^{\sigma}-p(t)u_2^{\sigma}\right)v_2^{\sigma}\right]\Delta t$$

$$-\int_{\rho(a)}^{b} \left[\left(v_2^{\Delta}+p(t)v_1^{\sigma}+q(t)v_2^{\sigma}\right)u_1^{\sigma}+\left(-v_1^{\Delta}+q(t)v_1^{\sigma}-p(t)v_2^{\sigma}\right)u_2^{\sigma}\right]\Delta t$$

$$= \int_{\rho(a)}^{b} \left\{u_2^{\Delta}v_1^{\sigma}+p(t)u_1^{\sigma}v_1^{\sigma}+q(t)u_2^{\sigma}v_1^{\sigma}-u_1^{\Delta}v_2^{\sigma}+q(t)u_1^{\sigma}v_2^{\sigma}-p(t)u_2^{\sigma}v_2^{\sigma}\right\}\Delta t$$

$$+\int_{\rho(a)}^{b} \left\{-v_2^{\Delta}u_1^{\sigma}-p(t)v_1^{\sigma}u_1^{\sigma}-q(t)v_2^{\sigma}u_1^{\sigma}+v_1^{\Delta}u_2^{\sigma}-q(t)v_1^{\sigma}u_2^{\sigma}+p(t)v_2^{\sigma}u_2^{\sigma}\right\}\Delta t$$

$$= \int_{\rho(a)}^{b} \left[u_2^{\Delta}v_1^{\sigma}-u_1^{\Delta}v_2^{\sigma}-v_2^{\Delta}u_1^{\sigma}+v_1^{\Delta}u_2^{\sigma}\right]\Delta t$$

$$= \int_{\rho(a)}^{b} \left\{W(u,v)\right\}^{\Delta}\Delta t = W(u,v)(b)-W(u,v)(\rho(a)).$$

In classical spectral theory, these are known as Lagrange's identity and Green's formula, respectively.

Theorem 3.11. [**20**] The equality

$$u_2(t,\lambda)\frac{\partial}{\partial\lambda}u_1(t,\lambda)-u_1(t,\lambda)\frac{\partial}{\partial\lambda}u_2(t,\lambda) = -\int_{\rho(a)}^{t}\left\{[u_1^{\sigma}(\tau,\lambda)]^2+[u_2^{\sigma}(\tau,\lambda)]^2\right\}\Delta\tau,$$

holds for all $t\in[\rho(a),b]\cap\mathbb{T}$ and $\lambda\in\mathbb{R}$.

Proof: Let v, $\lambda\in\mathbb{R}$ with $v\neq\lambda$. Let us use following equality

$$\begin{aligned}\{u_1(t,\lambda)u_2^{\sigma}(t,v)-u_1(t,v)u_2^{\sigma}(t,\lambda)\}^{\Delta} &= u_1^{\Delta}(t,\lambda)u_2(t,v)+u_1^{\sigma}(t,\lambda)u_2^{\Delta}(t,v)-u_1^{\Delta}(t,v)u_2(t,\lambda)\\ &\quad -u_1^{\sigma}(t,v)u_2^{\Delta}(t,\lambda).\end{aligned}$$

Then, we get

$$\begin{aligned}&\{u_1(t,\lambda)u_2(t,v)-u_1(t,v)u_2(t,\lambda)\}^{\Delta} =\\ =\ &[qu_1^{\sigma}(t,\lambda)-pu_2^{\sigma}(t,\lambda)-\lambda u_2^{\sigma}(t,\lambda)]u_2(t,v)+[-pu_1^{\sigma}(t,v)-qu_2^{\sigma}(t,v)+vu_1^{\sigma}(t,v)]u_1^{\sigma}(t,\lambda)\\ &-[qu_1^{\sigma}(t,v)-pu_2^{\sigma}(t,v)-vu_2^{\sigma}(t,v)]u_2(t,\lambda)-[-pu_1^{\sigma}(t,\lambda)-qu_2^{\sigma}(t,\lambda)+\lambda u_1^{\sigma}(t,\lambda)]u_1^{\sigma}(t,v)\\ =\ &qu_1^{\sigma}(t,\lambda)u_2(t,v)-pu_2^{\sigma}(t,\lambda)u_2(t,v)-\lambda u_2^{\sigma}(t,\lambda)u_2(t,v)-pu_1^{\sigma}(t,v)u_1^{\sigma}(t,\lambda)\\ &-qu_2^{\sigma}(t,v)u_1^{\sigma}(t,\lambda)+vu_1^{\sigma}(t,v)u_1^{\sigma}(t,\lambda)-qu_1^{\sigma}(t,v)u_2(t,\lambda)+pu_2^{\sigma}(t,v)u_2(t,\lambda)\\ &+vu_2^{\sigma}(t,v)u_2(t,\lambda)+pu_1^{\sigma}(t,\lambda)u_1^{\sigma}(t,v)+qu_2^{\sigma}(t,\lambda)u_1^{\sigma}(t,v)-\lambda u_1^{\sigma}(t,\lambda)u_1^{\sigma}(t,v).\\ =\ &(v-\lambda)\left(u_1^{\sigma}(t,v)u_1^{\sigma}(t,\lambda)+u_2^{\sigma}(t,\lambda)u_2(t,v)\right).\end{aligned}$$

Dividing both sides of above equality by $\lambda - v$ and taking limit as $v \to \lambda$, we have

$$\lim_{v\to\lambda} \frac{\{u_1(t,\lambda)\, u_2(t,v) - u_1(t,v)\, u_2(t,\lambda)\}^{\Delta}}{\lambda - v} = -\lim_{v\to\lambda} \{u_1^{\sigma}(t,\lambda)\, u_1^{\sigma}(t,v) + u_2^{\sigma}(t,\lambda)\, u_2^{\sigma}(t,v)\}$$

$$\Rightarrow \left\{u_2(t,\lambda)\frac{\partial}{\partial\lambda}u_1(t,\lambda) - u_1(t,\lambda)\frac{\partial}{\partial\lambda}u_2(t,\lambda)\right\}^{\Delta} = -\left\{[u_1^{\sigma}(t,\lambda)]^2 + [u_2^{\sigma}(t,\lambda)]^2\right\}.$$

By taking $\Delta-$integral of the last equality from $\rho(a)$ to t, we get

$$\int_{\rho(a)}^{t} \left\{u_2(\tau,\lambda)\frac{\partial}{\partial\lambda}u_1(\tau,\lambda) - u_1(\tau,\lambda)\frac{\partial}{\partial\lambda}u_2(\tau,\lambda)\right\}^{\Delta} \Delta\tau = -\int_{\rho(a)}^{t} \left\{[u_1^{\sigma}(\tau,\lambda)]^2 + [u_2^{\sigma}(\tau,\lambda)]^2\right\}\Delta\tau.$$

Since $u_1(\rho(a),\lambda) = \beta$ and $u_2(\rho(a),\lambda) = -\alpha$, it yields $\frac{\partial}{\partial\lambda}u_1(\rho(a),\lambda) = 0$ and $\frac{\partial}{\partial\lambda}u_2(\rho(a),\lambda) = 0$. Finally, after some computations, we obtain

$$u_2(t,\lambda)\frac{\partial}{\partial\lambda}u_1(t,\lambda) - u_1(t,\lambda)\frac{\partial}{\partial\lambda}u_2(t,\lambda) = -\int_{\rho(a)}^{t} \left\{[u_1^{\sigma}(\tau,\lambda)]^2 + [u_2^{\sigma}(\tau,\lambda)]^2\right\}\Delta\tau.$$

So, the proof is complete.

Theorem 3.12. [20] The component eigenfunctions $y_1(t,t_0)$ and $y_2(t,t_0)$ of the problem (3.3.1)-(3.3.3) have the following asymptotic estimates

$$\begin{aligned} y_1(t,t_0) &= -\alpha\sin_\lambda(t,\rho(a)) + \beta\cos_\lambda(t,\rho(a)) \\ &\quad + \int_{\rho(a)}^{t} \{[-p(s)\sin_\lambda(t,s) + q(s)\cos_\lambda(t,s)]\, y_1(s,t_0) \\ &\quad + [-p(s)\cos_\lambda(t,s) - q(s)\sin_\lambda(t,s)]\, y_2(s,t_0)\}\,\Delta s, \end{aligned}$$

and

$$\begin{aligned} y_2(t,t_0) &= -\alpha\cos_\lambda(t,\rho(a)) - \beta\sin_\lambda(t,\rho(a)) \\ &\quad + \int_{\rho(a)}^{t} \{[-p(s)\cos_\lambda(t,s) - q(s)\sin_\lambda(t,s)]\, y_1(s,t_0) \\ &\quad + [p(s)\sin_\lambda(t,s) - q(s)\cos_\lambda(t,s)]\, y_2(s,t_0)\}\,\Delta s, \end{aligned}$$

where $t, t_0 \in [\rho(a), b] \cap \mathbb{T}$, $y(t,t_0) = \begin{pmatrix} y_1(t,t_0), & y_2(t,t_0) \end{pmatrix}^T$.

Proof: Let us consider the system (3.3.4). Firstly, we need to find the homogeneous solution of this system. By homogeneous form of (3.3.4), one can easily get

$$y_2^{\Delta\Delta} + \lambda^2 y_2 = 0.$$

The homogeneous solution of above equation is

$$y_2(t,t_0) = c_1 \cos_\lambda(t,t_0) + c_2 \sin_\lambda(t,t_0). \tag{3.3.8}$$

Since $y_2^\Delta = \lambda y_1$, we have

$$y_1(t,t_0) = -c_1 \sin_\lambda(t,t_0) + c_2 \cos_\lambda(t,t_0), \tag{3.3.9}$$

where c_1 and c_2 are arbitrary constants. Now, we are ready to apply variation of parameters method (see [9]) to find particular solutions of the system (3.3.4). By (3.3.4), we can have

$$-c_1^\Delta \sin_\lambda(t,t_0) + c_2^\Delta \cos_\lambda(t,t_0) = -p(x)\, y_1 - q(x)\, y_2,$$

$$c_1^\Delta \cos_\lambda(t,t_0) + c_2^\Delta \sin_\lambda(t,t_0) = q(x)\, y_1 - p(x)\, y_2.$$

By multiplying these equations by $-\sin_\lambda(t,t_0)$, $\cos_\lambda(t,t_0)$ and $\cos_\lambda(t,t_0), \sin_\lambda(t,t_0)$, respectively, we get

$$c_1^\Delta = -\left[-p(x)\, y_1 - q(x)\, y_2\right] \sin_\lambda(t,t_0) + \left[q(s)\, y_1 - p(s)\, y_2\right] \cos_\lambda(t,t_0) \tag{3.3.10}$$

$$c_2^\Delta = \left[-p(x)\, y_1 - q(x)\, y_2\right] \cos_\lambda(t,t_0) + \left[q(x)\, y_1 - p(x)\, y_2\right] \sin_\lambda(t,t_0). \tag{3.3.11}$$

Integrating these equalities and considering (3.3.8) and (3.3.9), we obtain

$$\begin{aligned}
y_1 &= -\sin_\lambda(t,t_0) \int_{\rho(a)}^{t} \left\{\left[p(s)\, y_1 + q(s)\, y_2\right] \sin_\lambda(s,t_0) + \left[q(s)\, y_1 - p(s)\, y_2\right] \cos_\lambda(s,t_0)\right\} \Delta s \\
&\quad + \cos_\lambda(t,t_0) \int_{\rho(a)}^{t} \left\{\left[-p(s)\, y_1 - q(s)\, y_2\right] \cos_\lambda(s,t_0) + \left[q(s)\, y_1 - p(s)\, y_2\right] \sin_\lambda(s,t_0)\right\} \Delta s \\
&= \int_{\rho(a)}^{t} \{ p(s)\left[-\sin_\lambda(t,t_0)\sin_\lambda(s,t_0) - \cos_\lambda(t,t_0)\cos_\lambda(s,t_0)\right] y_1 \\
&\quad + q(s)\left[-\sin_\lambda(t,t_0)\cos_\lambda(s,t_0) + \cos_\lambda(t,t_0)\sin_\lambda(s,t_0)\right] y_1 \\
&\quad + p(s)\left[\sin_\lambda(t,t_0)\cos_\lambda(s,t_0) - \cos_\lambda(t,t_0)\sin_\lambda(s,t_0)\right] y_2 \\
&\quad + q(s)\left[-\sin_\lambda(t,t_0)\sin_\lambda(s,t_0) - \cos_\lambda(t,t_0)\cos_\lambda(s,t_0)\right] y_2 \} \Delta s \\
&= \int_{\rho(a)}^{t} \left\{\left[-p(s)\cos_\lambda(t,s) - q(s)\sin_\lambda(t,s)\right] y_1 + \left[p(s)\sin_\lambda(t,s) - q(s)\cos_\lambda(t,s)\right] y_2\right\} \Delta s,
\end{aligned}$$

then

$$y_1 = \int_{\rho(a)}^{t} \left\{\left(y_2^\Delta - \lambda y_1\right)\cos_\lambda(t,s) - \left(y_1^\Delta + \lambda y_2\right)\sin_\lambda(t,s)\right\} \Delta s, \tag{3.3.12}$$

and

$$
\begin{aligned}
y_2 &= \cos_\lambda(t,t_0)\int_{\rho(a)}^{t}\{[p(s)\,y_1+q(s)\,y_2]\sin_\lambda(s,t_0)+[q(s)\,y_1-p(s)\,y_2]\cos_\lambda(s,t_0)\}\,\Delta s\\
&\quad+\sin_\lambda(t,t_0)\int_{\rho(a)}^{t}\{[-p(s)\,y_1-q(s)\,y_2]\cos_\lambda(s,t_0)+[q(s)\,y_1-p(s)\,y_2]\sin_\lambda(s,t_0)\}\,\Delta s\\
&= \int_{\rho(a)}^{t}\{p(s)\,[\cos_\lambda(t,t_0)\sin_\lambda(s,t_0)-\sin_\lambda(t,t_0)\cos_\lambda(s,t_0)]\,y_1\\
&\quad+q(s)\,[\cos_\lambda(t,t_0)\cos_\lambda(s,t_0)+\sin_\lambda(t,t_0)\sin_\lambda(s,t_0)]\,y_1\\
&\quad+p(s)\,[-\cos_\lambda(t,t_0)\cos_\lambda(s,t_0)-\sin_\lambda(t,t_0)\sin_\lambda(s,t_0)]\,y_2\\
&\quad+q(s)\,[\cos_\lambda(t,t_0)\sin_\lambda(s,t_0)-\sin_\lambda(t,t_0)\cos_\lambda(s,t_0)]\,y_2\}\Delta s\\
&= \int_{\rho(a)}^{t}\{[-p(s)\sin_\lambda(t,s)+q(s)\cos_\lambda(t,s)]\,y_1+[-p(s)\cos_\lambda(t,s)-q(s)\sin_\lambda(t,s)]\,y_2\}\,\Delta s\\
&= \int_{\rho(a)}^{t}\{(-p(s)\,y_1-q(s)\,y_2)\sin_\lambda(t,s)+(q(s)\,y_1-p(s)\,y_2)\cos_\lambda(t,s)\}\,\Delta s
\end{aligned}
$$

then

$$
y_2=\int_{\rho(a)}^{t}\left\{\left(y_2^\Delta-\lambda y_1\right)\sin_\lambda(t,s)+\left(y_1^\Delta+\lambda y_2\right)\cos_\lambda(t,s)\right\}\Delta s. \tag{3.3.13}
$$

After using integration by parts in (3.3.12) and (3.3.13), respectively, we get

$$
\begin{aligned}
y_1 &= \int_{\rho(a)}^{t} y_2^\Delta\cos_\lambda(t,s)\,\Delta s-\int_{\rho(a)}^{t} y_1^\Delta\sin_\lambda(t,s)\,\Delta s-\lambda\int_{\rho(a)}^{t} y_1\cos_\lambda(t,s)\,\Delta s-\lambda\int_{\rho(a)}^{t} y_2\sin_\lambda(t,s)\,\Delta s\\
&= y_2(t,t_0)+\alpha\cos_\lambda(t,\rho(a))+\lambda\int_{\rho(a)}^{t} y_2^\sigma\sin_\lambda(t,s)\,\Delta s+\beta\sin_\lambda(t,\rho(a))\\
&\quad+\lambda\int_{\rho(a)}^{t} y_1^\sigma\cos_\lambda(t,s)\,\Delta s-\lambda\int_{\rho(a)}^{t} y_1\cos_\lambda(t,s)\,\Delta s-\lambda\int_{\rho(a)}^{t} y_2\sin_\lambda(t,s)\,\Delta s,
\end{aligned}
$$

or

$$
y_1=y_2(t,t_0)+\alpha\cos_\lambda(t,\rho(a))+\beta\sin_\lambda(t,\rho(a)), \tag{3.3.14}
$$

and

$$
y_2=\int_{\rho(a)}^{t} y_2^\Delta\sin_\lambda(t,s)\,\Delta s+\int_{\rho(a)}^{t} y_1^\Delta\cos_\lambda(t,s)\,\Delta s-\lambda\int_{\rho(a)}^{t} y_1\sin_\lambda(t,s)\,\Delta s+\lambda\int_{\rho(a)}^{t} y_2\cos_\lambda(t,s)\,\Delta s
$$

$$= \alpha \sin_\lambda (t, \rho(a)) - \lambda \int\limits_{\rho(a)}^{t} y_2^\sigma \cos_\lambda (t,s) \Delta s + y_1 (t, t_0) - \beta \cos_\lambda (t, \rho(a))$$
$$+\lambda \int\limits_{\rho(a)}^{t} y_1^\sigma \sin_\lambda (t,s) \Delta s - \lambda \int\limits_{\rho(a)}^{t} y_1 \sin_\lambda (t,s) \Delta s + \lambda \int\limits_{\rho(a)}^{t} y_2 \cos_\lambda (t,s) \Delta s,$$

or

$$y_2 = \alpha \sin_\lambda (t, \rho(a)) - \beta \cos_\lambda (t, \rho(a)) + y_1 (t, t_0). \tag{3.3.15}$$

Therefore, by considering (3.3.12), (3.3.14) and (3.3.13), (3.3.15), we have

$$\begin{aligned} y_1(t,t_0) &= -\alpha \sin_\lambda (t, \rho(a)) + \beta \cos_\lambda (t, \rho(a)) \\ &+ \int\limits_{\rho(a)}^{t} \{[-p(s) \sin_\lambda(t,s) + q(s) \cos_\lambda(t,s)]\, y_1(s,t_0) \\ &+ [-p(s) \cos_\lambda(t,s) - q(s) \sin_\lambda(t,s)]\, y_2(s,t_0)\} \,\Delta s, \end{aligned}$$

and

$$\begin{aligned} y_2(t,t_0) &= -\alpha \cos_\lambda (t, \rho(a)) - \beta \sin_\lambda (t, \rho(a)) \\ &+ \int\limits_{\rho(a)}^{t} \{[-p(s) \cos_\lambda(t,s) - q(s) \sin_\lambda(t,s)]\, y_1(s,t_0) \\ &+ [p(s) \sin_\lambda(t,s) - q(s) \cos_\lambda(t,s)]\, y_2(s,t_0)\} \,\Delta s, \end{aligned}$$

This completes proof.

Dirac eigenvalue problems appear in many fields and have several applications in mathematical physics. We notice that much attention paid on studying the eigenfunctions for this type problems for different operators on time scales. Hence, we have considered estimates of eigenfunctions for Dirac eigenvalue problem on an arbitrary time scale and generalized basic results. We have obtained some good results to fill the gaps in this area. We hope, this study will give a lead to the researchers.

4 IMPULSIVE DIFFUSION EQUATION ON TIME SCALES

In this chapter, we want to state some major spectral properties of impulsive diffusion eigenvalue problem of the form

$$Ly = -y^{\Delta\Delta}(t) + \{q(t) + 2\lambda p(t)\} y^{\sigma}(t) = \lambda^2 \varpi(t) y^{\sigma}(t), t \in [\rho(a), b] \cap \mathbb{T}, \tag{4.1}$$

with the boundary conditions

$$\begin{aligned} y^{\Delta}(\rho(a)) &= 0, & (4.2) \\ \alpha y(b) + \beta y^{\Delta}(b) &= 0, & (4.3) \end{aligned}$$

where λ is a spectral parameter and

$$\varpi(t) = \begin{cases} 1, & \rho(a) < t < \frac{b}{2} \\ \vartheta^2, & \frac{b}{2} < t < b \end{cases}, 0 < \vartheta \neq 1.$$

Here, we assume that $p, q : [\rho(a), b] \cap \mathbb{T} \to \mathbb{R}$ are continuous functions; $a, b \in \mathbb{T}$ with $a < b$, $y^{\sigma} = y(\sigma)$ and $\left(\alpha^2 + \beta^2\right) \neq 0$. $y(t)$ is known as eigenfunction of the problem (4.1)-(4.3). By taking $\mathbb{T} = \mathbb{R}$ in (4.1), we obtain below classical impulsive diffusion equation (or quadratic pencil of Schrödinger equation)

$$-y''(t) + \{q(t) + 2\lambda p(t)\} y(t) = \lambda^2 \varpi(t) y(t). \tag{4.4}$$

(4.4) is reduced to Sturm-Liouville equation when $p(t) = 0$ and $\varpi(t) = 1$ [42]. Now, we give some information related to the physical meaning and historical development of diffusion equation in classical spectral theory.

The issue of defining the interactions between colliding particles is very attractive in mathematical physics. In many cases, a definition can be carried out through a well known theoretical model. Notedly, one pays attention to the collisions of two spinless particles, and it is assumed that the $s-$ wave binding energies and $s-$wave scattering matrix are exactly known from collision experiments. $s-$wave Schrödinger equation with a radial static potential V can be given as

$$y'' + [E - V(x)] y = 0, \; x \geq 0, \tag{4.5}$$

where V depends on energy and has the below form

$$V(x, E) = U(x) + 2\sqrt{E} Q(x). \tag{4.6}$$

$U(x)$ and $Q(x)$ are complex-valued functions. (4.5) converts to the Klein-Gordon $s-$wave equation with the static potential $Q(x)$, for a particle of zero mass and the energy $\sqrt{E}$ with the supplementary term $U(x) = -Q^2(x)$ [43]. The Klein Gordon equation is considered one of the most important mathematical models in quantum field theory. This equation appears in relativistic physics and is used to describe dispersive wave phenomena in general [44].

Jaulent and Jean [43] expressed the actual background of diffusion operator in 1972. Maksudov and Guseinov [45] considered the spectral theory of diffusion operator in 1979. Then, Gasymov and Guseinov [46] determined the diffusion operator from spectral data in 1981. Fragela [47] studied on this operator with integral boundary conditions in 1984. Maksudov and Guseinov [48] gave some solutions of inverse scattering problem for quadratic pencil of Schrödinger operator on whole axis in 1986. Bairamov, Çakar and Celebi [49] obtained some results for diffusion operator with spectral singularities, discrete spectrum and principal functions in 1997. Moreover, inverse nodal problem was first considered by Koyunbakan [50] in 2007 for diffusion operator. Koyunbakan [51], [52] obtained uniqueness results and reconstruction formulas in 2008 with Panakhov. Yilmaz and Koyunbakan [53] expressed reconstruction formulas for the $k-th$ derivatives of the first potential function q in diffusion equation by using nodal points in 2008. Yang [54] extended their results to the second potential function p in 2010. Yang and Zettl [55] solved half-inverse diffusion problem in 2012. We can replicate these examples. There are many other researches on diffusion operator in literature by many authors (see [56], [57], [58] , [59], [60], [61], [62], [63], [64], [65], [66]).

This chapter is arranged as follows: In Section 4.1, we prove some basic theorems for impulsive diffusion equation on $\mathbb{T}$. Using some methods, we get asymptotic estimate of eigenfunction for the problem (4.1)-(4.3) in Section 4.2.

4.1 Some spectral properties of Impulsive Diffusion Equation on time scales

It is well known that the problem (4.1)-(4.3) has only real, simple eigenvalues and its eigenfunctions are orthogonal when $\mathbb{T} = \mathbb{R}$ [46]. The below results generalize this basic consequences to $\mathbb{T}$.

Theorem 4.1. The eigenvalues of the problem (4.1)-(4.3) are all simple.

Proof: Let $\lambda \in \mathbb{R}$ be spectral parameter and $y_1(t)$, $y_2(t)$ be eigenfunctions of (4.1)-(4.3). Then, we have

$$Ly_1 = -y_1^{\Delta\Delta}(t) + \{q(t) + 2\lambda p(t)\}\, y_1^{\sigma}(t) = \lambda^2 \varpi(t) y_1^{\sigma}(t),$$

$$Ly_2 = -y_2^{\Delta\Delta}(t) + \{q(t) + 2\lambda p(t)\}\, y_2^{\sigma}(t) = \lambda^2 \varpi(t) y_2^{\sigma}(t).$$

Multiplying these equations by y_2^{σ} and y_1^{σ} respectively and substracting, we get

$$\left(y_1 y_2^{\Delta} - y_2 y_1^{\Delta}\right)^{\Delta} = 0. \tag{4.1.1}$$

By taking $\Delta-$integral of (4.1.1) on $[\rho(a), b]$, we conclude

$$W[y_1, y_2]\Big|_{\rho(a)}^{b} = c.$$

Using the conditions (4.2), (4.3), one can easily obtain that $c = 0$. Then

$$\left(\frac{y_1}{y_2}\right)^{\Delta} = 0.$$

The last equality means that $y_1 = ky_2$ (k constant) and it implies y_1 and y_2 are linearly dependent solutions of (4.1)-(4.3). This completes the proof.

Theorem 4.2. All eigenvalues of the problem (4.1)-(4.3) are real.

Proof: Let $\overline{\lambda}_0$ be a complex eigenvalue and $\overline{y}(t)$ be an eigenfunction of the problem (4.1)-(4.3) corresponding to the eigenvalue $\overline{\lambda}_0$. Then, we obtain

$$\begin{aligned}
\left\{y^{\Delta}\overline{y} - \overline{y}^{\Delta}y\right\}^{\Delta} &= y^{\Delta\Delta}\overline{y}^{\sigma} - \overline{y}^{\Delta\Delta}y^{\sigma} \\
&= \left\{q(t) + 2\lambda_0 p(t) - \lambda_0^2\varpi(t)\right\} y^{\sigma}\overline{y}^{\sigma} - \left\{q(t) + 2\overline{\lambda}_0 p(t) - \overline{\lambda}_0^2\varpi(t)\right\} \overline{y}^{\sigma}y^{\sigma} \\
&= \left(\overline{\lambda}_0^2 - \lambda_0^2\right)\varpi(t)y^{\sigma}\overline{y}^{\sigma} - 2\left(\overline{\lambda}_0 - \lambda_0\right)p(t)y^{\sigma}\overline{y}^{\sigma} \\
&= \left(\overline{\lambda}_0^2 - \lambda_0^2\right)\varpi(t)\left|y^{\sigma}(t)\right|^2 - 2\left(\overline{\lambda}_0 - \lambda_0\right)p(t)\left|y^{\sigma}(t)\right|^2.
\end{aligned}$$

If we take $\Delta-$integral of the last equality from $\rho(a)$ to b, we get

$$\begin{aligned}
\left(\overline{\lambda}_0^2 - \lambda_0^2\right)\int_{\rho(a)}^{b}\varpi(t)\left|y^{\sigma}(t)\right|^2\Delta t - 2\left(\overline{\lambda}_0 - \lambda_0\right)\int_{\rho(a)}^{b}p(t)\left|y^{\sigma}(t)\right|^2\Delta t &= y^{\Delta}(b)\overline{y}(b) - \overline{y}^{\Delta}(b)y(b) \\
&\quad -y^{\Delta}(\rho(a))\overline{y}(\rho(a)) + \overline{y}^{\Delta}(\rho(a))y(\rho(a)) \\
&= 0,
\end{aligned}$$

by considering the conditions (4.2), (4.3). So, we have

$$\left(\overline{\lambda}_0 + \lambda_0\right)\int_{\rho(a)}^{b}\varpi(t)\left|y^{\sigma}(t)\right|^2\Delta t - 2\int_{\rho(a)}^{b}p(t)\left|y^{\sigma}(t)\right|^2\Delta t = 0 \Rightarrow y^{\sigma}(t) = 0\ ,$$

where $\left(\overline{\lambda}_0 + \lambda_0\right)\varpi(t) - 2p(t) > 0$ for $\overline{\lambda}_0 \neq \lambda_0$. This is a contradiction. Hence, this completes proof.

Theorem 4.3. Let $y_1, y_2 \in C_{rd}^1(\mathbb{T})$ be the eigenfunctions of the problem (4.1)-(4.3). Then,

a) $(Ly_1)\,y_2^{\sigma} - (Ly_2)\,y_1^{\sigma} = W^{\Delta}(y_1, y_2)$ on $[\rho(a), b] \cap \mathbb{T}$.

b) $< Ly_1, y_2^{\sigma} > - < Ly_2, y_1^{\sigma} >= W(y_1, y_2)(b) - W(y_1, y_2)(\rho(a))$,

where $W(y_1, y_2) = y_1y_2^{\Delta} - y_2y_1^{\Delta}$ is the Wronskian of the functions y_1 and y_2.

Proof: a) Definition of the Wronskian and product rule for $\Delta-$derivative give following result:

$$\begin{aligned} W^{\Delta}(y_1,y_2) &= y_2^{\Delta\Delta}y_1^{\sigma} - y_1^{\Delta\Delta}y_2^{\sigma} \\ &= -y_1^{\sigma}\left(-y_2^{\Delta\Delta} + (q(t)+2\lambda p(t))\,y_2^{\sigma}\right) + y_2^{\sigma}\left(-y_1^{\Delta\Delta} + (q(t)+2\lambda p(t))\,y_1^{\sigma}\right) \\ &= (Ly_1)\,y_2^{\sigma} - (Ly_2)\,y_1^{\sigma}. \end{aligned}$$

b) By using definition of Wronskian and inner product on the set of so-called $rd-$continuous functions, we have

$$< \ Ly_1, y_2^{\sigma} > - < Ly_2, y_1^{\sigma} >= \int\limits_{\rho(a)}^{b} \left[(Ly_1)\,y_2^{\sigma} - (Ly_2)\,y_1^{\sigma}\right]\Delta t$$

$$\begin{aligned} &= \int\limits_{\rho(a)}^{b} \left\{\left(-y_1^{\Delta\Delta} + (q(t)+2\lambda p(t))\,y_1^{\sigma}\right)y_2^{\sigma} - \left(-y_2^{\Delta\Delta} + (q(t)+2\lambda p(t))\,y_2^{\sigma}\right)y_1^{\sigma}\right\}\Delta t \\ &= -\int\limits_{\rho(a)}^{b} \left\{y_1^{\Delta\Delta}y_2^{\sigma} - y_2^{\Delta\Delta}y_1^{\sigma}\right\}\Delta t \\ &= \int\limits_{\rho(a)}^{b} \left\{W(y_1,y_2)\right\}^{\Delta}\Delta t \\ &= W(y_1,y_2)(b) - W(y_1,y_2)(\rho(a)). \end{aligned}$$

Hence, this completes the proof. In literature, these equalities are known as Lagrange's identity and Green's formula, respectively.

Theorem 4.4. The equality

$$y\,(t,\lambda)\,\frac{\partial}{\partial\lambda}y^{\Delta}\,(t,\lambda) - y^{\Delta}\,(t,\lambda)\,\frac{\partial}{\partial\lambda}y\,(t,\lambda) = 2\lambda\int\limits_{\rho(a)}^{t}\varpi(\tau)\,(y^{\sigma}\,(\tau,\lambda))^2\,\Delta\tau - 2\int\limits_{\rho(a)}^{t}p(\tau)\,(y^{\sigma}\,(\tau,\lambda))^2\,\Delta\tau,$$

holds for all $t\in[\rho(a),b]\cap\mathbb{T}$ and $\lambda\in\mathbb{R}$.

Proof: Let γ, $\lambda\in\mathbb{R}$ with $\gamma\neq\lambda$. Then,

$$\begin{aligned} \left\{y\,(t,\gamma)\,y^{\Delta}\,(t,\lambda) - y^{\Delta}\,(t,\gamma)\,y\,(t,\lambda)\right\}^{\Delta} &= y^{\sigma}\,(t,\gamma)\,y^{\Delta\Delta}\,(t,\lambda) - y^{\Delta\Delta}\,(t,\gamma)\,y^{\sigma}\,(t,\lambda) \\ &= y^{\sigma}\,(t,\gamma)\left\{q(t)+2\lambda p(t)-\lambda^2\varpi(t)\right\}y^{\sigma}\,(t,\lambda) \\ &\quad -y^{\sigma}\,(t,\lambda)\left\{q(t)+2\gamma p(t)-\gamma^2\varpi(t)\right\}y^{\sigma}\,(t,\gamma) \\ &= \left(\gamma^2-\lambda^2\right)\varpi(t)y^{\sigma}(t,\lambda)y^{\sigma}(t,\gamma) - 2\,(\gamma-\lambda)\,p(t)y^{\sigma}(t,\lambda)y^{\sigma}(t,\gamma). \end{aligned}$$

Dividing both sides of above equality by $\lambda-\gamma$ and taking limit as $\gamma\to\lambda$, we have

$$\lim_{\gamma\to\lambda}\frac{\left\{y\left(t,\gamma\right)y^{\Delta}\left(t,\lambda\right)-y^{\Delta}\left(t,\gamma\right)y\left(t,\lambda\right)\right\}^{\Delta}}{\lambda-\gamma}=-\lim_{\gamma\to\lambda}\left\{\left(\gamma+\lambda\right)\varpi(t)y^{\sigma}(t,\lambda)y^{\sigma}(t,\gamma)-2p(t)y^{\sigma}(t,\lambda)y^{\sigma}(t,\gamma)\right\}$$

and

$$\left\{y\left(t,\lambda\right)\frac{\partial}{\partial\lambda}y^{\Delta}\left(t,\lambda\right)-y^{\Delta}\left(t,\lambda\right)\frac{\partial}{\partial\lambda}y\left(t,\lambda\right)\right\}^{\Delta}=-2\lambda\varpi(t)\left(y^{\sigma}\left(t,\lambda\right)\right)^{2}+2p(t)\left(y^{\sigma}\left(t,\lambda\right)\right)^{2}.$$

By taking $\Delta-$integral of the last equality from $\rho(a)$ to t, we get

$$\begin{aligned}\int_{\rho(a)}^{t}\left\{y\left(\tau,\lambda\right)\frac{\partial}{\partial\lambda}y^{\Delta}\left(\tau,\lambda\right)-y^{\Delta}\left(\tau,\lambda\right)\frac{\partial}{\partial\lambda}y\left(\tau,\lambda\right)\right\}^{\Delta}\Delta\tau &= -2\lambda\int_{\rho(a)}^{t}\varpi(\tau)\left(u^{\sigma}\left(\tau,\lambda\right)\right)^{2}\Delta\tau\\ &\quad +2\int_{\rho(a)}^{t}p(\tau)\left(u^{\sigma}\left(\tau,\lambda\right)\right)^{2}\Delta\tau.\end{aligned}$$

Since $y\left(\rho(a),\lambda\right)=1$ and $y^{\Delta}\left(\rho(a),\lambda\right)=0$, it yields $\frac{\partial}{\partial\lambda}y\left(\rho(a),\lambda\right)=0$ and $\frac{\partial}{\partial\lambda}y^{\Delta}\left(\rho(a),\lambda\right)=0$. Finally, after some computations, we obtain

$$y\left(t,\lambda\right)\frac{\partial}{\partial\lambda}y^{\Delta}\left(t,\lambda\right)-y^{\Delta}\left(t,\lambda\right)\frac{\partial}{\partial\lambda}y\left(t,\lambda\right)=-2\lambda\int_{\rho(a)}^{t}\varpi(\tau)\left(u^{\sigma}\left(\tau,\lambda\right)\right)^{2}\Delta\tau+2\int_{\rho(a)}^{t}p(\tau)\left(u^{\sigma}\left(\tau,\lambda\right)\right)^{2}\Delta\tau.$$

So, the proof is completed.

Theorem 4.5. The eigenfunctions $y_1(t,\lambda_1)$ and $y_2(t,\lambda_2)$ of the problem (4.1)-(4.3) corresponding to distinct eigenvalues λ_1 and λ_2 are orthogonal , i.e

$$\left(\lambda_1+\lambda_2\right)\int_{\rho(a)}^{b}\varpi(t)y_1^{\sigma}(t,\lambda_1)y_2^{\sigma}(t,\lambda_2)\Delta t-2\int_{\rho(a)}^{b}p(t)y_1^{\sigma}(t,\lambda_1)y_2^{\sigma}(t,\lambda_2)\Delta t=0.$$

Proof: Let us use following equality

$$\left\{y_1^{\Delta}(t,\lambda_1)y_2(t,\lambda_2)-y_2^{\Delta}(t,\lambda_2)y_1(t,\lambda_1)\right\}^{\Delta}=\left\{\left(\lambda_2^2-\lambda_1^2\right)\varpi(t)-2\left(\lambda_2-\lambda_1\right)p(t)\right\}y_1^{\sigma}(t,\lambda_1)y_2^{\sigma}(t,\lambda_2).$$

Taking $\Delta-$integral of the last equality from $\rho(a)$ to b, we get

$$\begin{aligned}\int_{\rho(a)}^{b}\left\{y_1^{\Delta}(t,\lambda_1)y_2(t,\lambda_2)-y_2^{\Delta}(t,\lambda_2)y_1(t,\lambda_1)\right\}^{\Delta}\Delta t &= \left(\lambda_2^2-\lambda_1^2\right)\int_{\rho(a)}^{b}\varpi(t)y_1^{\sigma}(t,\lambda_1)y_2^{\sigma}(t,\lambda_2)\Delta t\\ &\quad -2\left(\lambda_2-\lambda_1\right)\int_{\rho(a)}^{b}p(t)y_1^{\sigma}(t,\lambda_1)y_2^{\sigma}(t,\lambda_2)\Delta t\\ &= y_1^{\Delta}(b,\lambda_1)y_2(b,\lambda_2)-y_2^{\Delta}(b,\lambda_2)y_1(b,\lambda_1)\\ &\quad -y_1^{\Delta}(\rho(a),\lambda_1)y_2(\rho(a),\lambda_2)+y_2^{\Delta}(\rho(a),\lambda_2)y_1(\rho(a),\lambda_1)\\ &= 0,\end{aligned}$$

and

$$(\lambda_1+\lambda_2)\int\limits_{\rho(a)}^{b}\varpi(t)y_1^{\sigma}(t,\lambda_1)y_2^{\sigma}(t,\lambda_2)\Delta t-2\int\limits_{\rho(a)}^{b}p(t)y_1^{\sigma}(t,\lambda_1)y_2^{\sigma}(t,\lambda_2)\Delta t=0,$$

for $\lambda_1\neq\lambda_2$. Then, it shows that the eigenfunctions $y_1(t,\lambda_1)$ and $y_2(t,\lambda_2)$ corresponding to distinct eigenvalues are always orthogonal.

4.2 Main Results on impulsive diffusion equation on time scales

In this section, we get asymptotic estimate for the eigenfunction $y(t,\lambda)$ of the problem (4.1)-(4.3) on $\mathbb{T}$ whose all points are right dense.

Theorem 4.6. The eigenfunction $y(t,\lambda)$ and its $\Delta-$derivative of the problem (4.1)-(4.3) have the following asymptotic estimates

$$y(t,\lambda)=\cos_{\lambda}(t,\rho(a))-\frac{1}{\lambda}\int\limits_{\rho(a)}^{t}\left(q(s)+2\lambda p(s)\right)y(s,\lambda)\sin_{\lambda}(t,s)\Delta s,$$

and

$$y^{\Delta}(t,\lambda)=-\lambda\sin_{\lambda}(t,\rho(a))-\int\limits_{\rho(a)}^{t}\left(q(s)+2\lambda p(s)\right)y(s,\lambda)\cos_{\lambda}(t,s)\Delta s,$$

where $t\in[\rho(a),b]\cap\mathbb{T}$.

Proof: By equation (4.1), we can write

$$\begin{aligned}\frac{1}{\lambda}\int\limits_{\rho(a)}^{t}\left(q(s)+2\lambda p(s)\right)y(s,\lambda)\sin_{\lambda}(t,s)\Delta s &= \frac{1}{\lambda}\int\limits_{\rho(a)}^{t}\sin_{\lambda}(t,s)\left(y^{\Delta\Delta}(s,\lambda)+\lambda^2\varpi(s)y(s,\lambda)\right)\Delta s\\ &= \frac{1}{\lambda}\int\limits_{\rho(a)}^{t}\sin_{\lambda}(t,s)y^{\Delta\Delta}(s,\lambda)\Delta s\\ &\quad+\lambda\int\limits_{\rho(a)}^{t}\sin_{\lambda}(t,s)\varpi(s)y(s,\lambda)\Delta s. \qquad (4.3.1)\end{aligned}$$

After using integration by parts twice for the first integral in (4.3.1.), we get

$$\frac{1}{\lambda}\int\limits_{\rho(a)}^{t}\sin_{\lambda}(t,s)y^{\Delta\Delta}(s,\lambda)\Delta s=\frac{1}{\lambda}\left(\sin_{\lambda}(t,s)y^{\Delta}(s,\lambda)\right)\Big|_{\rho(a)}^{t}-\frac{1}{\lambda}\int\limits_{\rho(a)}^{t}\lambda\cos_{\lambda}(t,s)y^{\Delta}(s,\lambda)\Delta s$$

$$\begin{aligned} &= \frac{1}{\lambda}\sin_\lambda(t,t)y^\Delta(t,\lambda) - \frac{1}{\lambda}\sin_\lambda(t,\rho(a))y^\Delta(\rho(a),\lambda) \\ &\quad - (\cos_\lambda(t,s)y(s,\lambda))\Big|_{\rho(a)}^{t} + \int_{\rho(a)}^{t} -\lambda\sin_\lambda(t,s)y(s,\lambda)\Delta s \\ &= -\frac{1}{\lambda}\sin_\lambda(t,\rho(a))y^\Delta(\rho(a),\lambda) - y(t,\lambda) \\ &\quad + \cos_\lambda(t,\rho(a))y(\rho(a),\lambda) - \lambda\int_{\rho(a)}^{t}\sin_\lambda(t,s)y(s,\lambda)\Delta s, \end{aligned}$$

and by the conditions (4.2) and (4.3), we get

$$\frac{1}{\lambda}\int_{\rho(a)}^{t}\sin_\lambda(t,s)y^{\Delta\Delta}(s,\lambda)\Delta s = -y(t,\lambda) + \cos_\lambda(t,\rho(a)) - \lambda\int_{\rho(a)}^{t}\sin_\lambda(t,s)y(s,\lambda)\Delta s. \tag{4.3.2}$$

Therefore, considering (4.3.1) and (4.3.2), we have

$$y(t,\lambda) = \cos_\lambda(t,\rho(a)) - \frac{1}{\lambda}\int_{\rho(a)}^{t}(q(s)+2\lambda p(s))\,y(s,\lambda)\sin_\lambda(t,s)\Delta s, \tag{4.3.3}$$

where $\varpi(t)=1$. Differentiating (4.3.3) with respect to t, we obtain

$$y^\Delta(t,\lambda) = -\lambda\sin_\lambda(t,\rho(a)) - \frac{1}{\lambda}\sin_\lambda(\sigma(t),t)(q(t)+2\lambda p(t))\,y(t,\lambda) - \frac{1}{\lambda}\int_{\rho(a)}^{t} w(s)y(s,\lambda)\lambda\cos_\lambda(t,\rho(a))\,\Delta s,$$

or since $\sin_\lambda(\sigma(t),t) = \sin_\lambda(t,t) = 0$, we have

$$y^\Delta(t,\lambda) = -\lambda\sin_\lambda(t,\rho(a)) - \int_{\rho(a)}^{t}(q(s)+2\lambda p(s))\,y(s,\lambda)\cos_\lambda(t,s)\Delta s. \tag{4.3.4}$$

This completes the proof.

Impulsive type diffusion eigenvalue problems appear in many fields of mathematical physics as quantum field theory, relativistics physics. Because of this importance, we notice that discussing the solutions of these type of problems on time scales will attract attention.

5 BESSEL EQUATION ON TIME SCALES

In this chapter, we try to prove some fundamental spectral properties of Bessel eigenvalue problem of the form

$$L_l u = -u^{\Delta\Delta}(t) + \left\{ w_0(t) + \frac{l(l+1)}{t^2} \right\} u^\sigma(t) = \lambda u^\sigma(t), t \in [\rho(a), b] \cap \mathbb{T}, \tag{5.1}$$

with the boundary conditions

$$u^\Delta(\rho(a)) = 0, \tag{5.2}$$

$$\gamma u(b) + \delta u^\Delta(b) = 0, \tag{5.3}$$

where λ is a spectral parameter and l is positive number. During this study, we will suppose that $w_0 : [\rho(a), b] \cap \mathbb{T} \to \mathbb{R}$ is continuous potential function; $a, b \in \mathbb{T}$ with $a < b$, $u^\sigma = u(\sigma)$ and $\left(\gamma^2 + \delta^2\right) \neq 0$. $u(t)$ is known as eigenfunction of the problem (5.1)-(5.3) and $L(l, w_0)$ or L_l denotes the Bessel operator. By taking $\mathbb{T} = \mathbb{R}$ in (5.1), we get the below classical Bessel equation

$$-u''(t) + \left\{ w_0(t) + \frac{l(l+1)}{t^2} \right\} u(t) = \lambda u(t), t \in [a, b]. \tag{5.4}$$

(5.4) is usually called Liouville form of Bessel equation since its eigenfunctions are expressed in terms of Bessel functions for $w_0 \equiv 0$ [67]. (5.4) is reduced to classical Sturm-Liouville equation when $l = 0$. The operator $L(l, w_0)$ is bounded below and has simple spectrum in classical case [68]. Now, we will give some information related to the historical development and physical meaning of Bessel equation in classical spectral theory. These type operators arise in spectral analysis of the radial Schrödinger operators $-\Delta + w_0(|x|)$ acting on a unit ball in Euclidean space. In fact, it is well known that the latter can be decomposed in spherical coordinates into the direct sum of operators L_l parametrized by the angular momentum $l \in \mathbb{Z}^+$ and given by

$$L_l = -\frac{d^2}{dt^2} + \frac{l(l+1)}{t^2} + w_0(t).$$

The first studies on the classical Bessel equations were given by Willson, Peirce [69] and Chessin [70]. Chessin obtained some important results about the general solution of the Bessel equation in 1894. Willson and Peirce gave a table of the first forty roots of Bessel equation in 1897. Then, Stashevskaya [71] studied inverse spectral problem for differential operators having singularity at zero in 1950's. For the case $l = 0$ in (1.4), there is comprehensive inverse spectral theory in literature. Notedly, Pöschel and Trubowitz [72] studied in detail the mapping sending the potential w_0 into the corresponding spectral data, and proved that this mapping is one to one and analytic in 1987. Then, Guillot and Ralston [73] extended the method of them to the case $l = 1$ in 1988. Serier [74] generalized this analysis to the case of an arbitrary $l \in \mathbb{N}$. The spectra of the operator $L(l, w_0)$ for an arbitrary $l \in \mathbb{N}$ were also completely characterized in 1993 by Carlson [75]. Zhornitskaya and Serov [76] gave some uniqueness results for Bessel operator in 1994. To develop the ideas of Pöschel and Trubowitz, Carlson [77] proved many results on the unique reconstruction of $L(l, w_0)$ from the spectral data for all real $l \geq -1/2$ in 1997. Gasymov [78] announced the solution of the inverse spectral problem for the Bessel operator $L(l, w_0)$ with $l \in \mathbb{N}$ in his short paper in 1965. His approach was based on the double commutation method which allows the reduction of the inverse problem to the classical case $l = 0$. There are many other studies about Bessel type operators in classical spectral theory by several authors (see [79], [80], [81], [82], [83], [84], [85], [86], [87]). Our aim is to define all these properties for the spectral theory of Bessel equation on $\mathbb{T}$.

This section is organized as follows: In subsection 5. 1., we prove some basic theorems for Bessel equation on $\mathbb{T}$. Using some techniques, we get asymptotic estimate of eigenfunction for the Bessel eigenvalue problem (5.1)-(5.3) in subsection 5.2. In subsection 5.3., we give a conclusion to sum up our study.

5.1 Some spectral properties of Bessel Equation on time scales

It is well known that the problem (5.1)-(5.3) has only real, simple eigenvalues and eigenfunctions are orthogonal in case of $\mathbb{T} = \mathbb{R}$ [68]. The below theorems extend this basic results into $\mathbb{T}$.

Theorem 5.1. The eigenvalues of the problem (5.1)-(5.3) are all simple.

Proof: Let $\lambda \in \mathbb{R}$ be a spectral parameter and $u(t)$, $v(t)$ be eigenfunctions of (5.1)-(5.3). Then, we have

$$L_l u = -u^{\Delta\Delta}(t) + \left\{ w_0(t) + \frac{l(l+1)}{t^2} \right\} u^\sigma(t) = \lambda u^\sigma(t),$$
$$L_l v = -v^{\Delta\Delta}(t) + \left\{ w_0(t) + \frac{l(l+1)}{t^2} \right\} v^\sigma(t) = \lambda v^\sigma(t).$$

Multiplying these equations by v^σ and u^σ respectively and substracting, we get

$$\left(uv^\Delta - vu^\Delta\right)^\Delta = 0. \tag{5.1.1}$$

By taking $\Delta-$integral of (5.1.1) on $[\rho(a), b]$, it yields

$$W[u, v] \Big|_{\rho(a)}^{b} = c.$$

Using the conditions (5.2) and (5.3), one can easily obtain $c = 0$. Then

$$\left(\frac{u}{v}\right)^\Delta = 0.$$

The last equality gives $u = kv$ (k constant) and it implies that u and v are linearly dependent solutions of (5.1)-(5.3). So, it completes the proof.

Theorem 5.2. All eigenvalues of the problem (5.1)-(5.3) are real.

Proof: Let $\overline{\lambda}_0$ be a complex eigenvalue and $\overline{u}(t)$ be an eigenfunction corresponding to the eigenvalue $\overline{\lambda}_0$ of the problem (5.1)-(5.3). Then, we obtain

$$\begin{aligned} \left\{u^{\Delta}\overline{u} - \overline{u}^{\Delta}u\right\}^{\Delta} &= u^{\Delta}\overline{u}^{\Delta} + u^{\Delta\Delta}\overline{u}^{\sigma} - \overline{u}^{\Delta}u^{\Delta} - \overline{u}^{\Delta\Delta}u^{\sigma} \\ &= \left(w(x) - \lambda_0\right) u^{\sigma}\overline{u}^{\sigma} - \left(w(x) - \overline{\lambda}_0\right) \overline{u}^{\sigma}u^{\sigma} \\ &= \left(\overline{\lambda}_0 - \lambda_0\right) \left|u^{\sigma}(t)\right|^2, \end{aligned}$$

where $w(t) = w_0(t) + \dfrac{l(l+1)}{t^2}$. If we take $\Delta-$integral of the last equality from $\rho(a)$ to b, we get

$$\begin{aligned} \left(\overline{\lambda}_0 - \lambda_0\right) \int\limits_{\rho(a)}^{b} \left|u^{\sigma}(t)\right|^2 \Delta t &= u^{\Delta}(b)\overline{u}(b) - \overline{u}^{\Delta}(b)u(b) - u^{\Delta}(\rho(a))\overline{u}(\rho(a)) + \overline{u}^{\Delta}(\rho(a))u(\rho(a)) \\ &= 0, \end{aligned}$$

by the conditions (5.2), (5.3). Therefore, we have

$$\int\limits_{\rho(a)}^{b} \left|u^{\sigma}(t)\right|^2 \Delta t = 0,$$

and

$$u^{\sigma}(t) = 0\ ,$$

for $\overline{\lambda}_0 \neq \lambda_0$. This is a contradiction. Hence, this contradiction completes the proof.

Theorem 5.3. Let $u, v \in C^1_{rd}(\mathbb{T})$ be the eigenfunctions of the problem (5.1)-(5.3). Then, we have the following equalities.

a) $(Lu)\, v^{\sigma} - (Lv)\, u^{\sigma} = W^{\Delta}(u, v)$ on $[\rho(a), b] \cap \mathbb{T}$.

b) $< Lu, v^{\sigma} > - < Lv, u^{\sigma} >= W(u, v)(b) - W(u, v)(\rho(a))$,

where $W(u, v) = uv^{\Delta} - vu^{\Delta}$ is the Wronskian of the functions u and v.

Proof:

a) Definition of the Wronskian and product rule for $\Delta-$derivative give the below result:

$$\begin{aligned} W^{\Delta}(u, v) &= \left(uv^{\Delta} - vu^{\Delta}\right)^{\Delta} \\ &= -u^{\sigma}\left(-v^{\Delta\Delta} + wv^{\sigma}\right) + v^{\sigma}\left(-u^{\Delta\Delta} + wu^{\sigma}\right) \\ &= (Lu)\, v^{\sigma} - (Lv)\, u^{\sigma}. \end{aligned}$$

b) By using definition of Wronskian and inner product on the set of so-called $rd-$continuous functions, we have

$$
\begin{aligned}
< \quad Lu, v^{\sigma} > - < Lv, u^{\sigma} > &= \int_{\rho(a)}^{b} \left[(Lu)\, v^{\sigma} - (Lv)\, u^{\sigma} \right] \Delta t \\
&= \int_{\rho(a)}^{b} \left\{ \left(-u^{\Delta\Delta} + wu^{\sigma} \right) v^{\sigma} - \left(-v^{\Delta\Delta} + wv^{\sigma} \right) u^{\sigma} \right\} \Delta t \\
&= -\int_{\rho(a)}^{b} \left\{ u^{\Delta\Delta} v^{\sigma} - v^{\Delta\Delta} u^{\sigma} \right\} \Delta t \\
&= \int_{\rho(a)}^{b} \left\{ W(u,v) \right\}^{\Delta} \Delta t \\
&= W(u,v)(b) - W(u,v)(\rho(a)).
\end{aligned}
$$

This completes the proof. In classical spectral theory, these equalities are known as Lagrange's identity and Green's formula, respectively.

Theorem 5.4. The equality

$$
u\,(t,\lambda)\,\frac{\partial}{\partial\lambda} u^{\Delta}\,(t,\lambda) - u^{\Delta}\,(t,\lambda)\,\frac{\partial}{\partial\lambda} u\,(t,\lambda) = -\int_{\rho(a)}^{t} \left(u^{\sigma}\,(\tau,\lambda) \right)^{2} \Delta\tau,
$$

holds for all $t \in [\rho(a), b] \cap \mathbb{T}$ and $\lambda \in \mathbb{R}$.

Proof: Let μ, $\lambda \in \mathbb{R}$ with $v \neq \lambda$. Then,

$$
\begin{aligned}
\left\{ u\,(t,\mu)\, u^{\Delta}\,(t,\lambda) - u^{\Delta}\,(t,\mu)\, u\,(t,\lambda) \right\}^{\Delta} &= u^{\Delta}\,(t,\mu)\, u^{\Delta}\,(t,\lambda) + u^{\sigma}\,(t,\mu)\, u^{\Delta\Delta}\,(t,\lambda) \\
&\quad - u^{\Delta}\,(t,\mu)\, u^{\Delta}\,(t,\lambda) - u^{\Delta\Delta}\,(t,\mu)\, u^{\sigma}\,(t,\lambda) \\
&= u^{\sigma}\,(t,\mu)\, u^{\Delta\Delta}\,(t,\lambda) - u^{\Delta\Delta}\,(t,\mu)\, u^{\sigma}\,(t,\lambda) \\
&= u^{\sigma}\,(t,\mu)\,(w(t)-\lambda)\, u^{\sigma}\,(t,\lambda) - u^{\sigma}\,(t,\lambda)\,(w(t)-\mu) u^{\sigma}\,(t,\mu) \\
&= (\mu-\lambda)\, u^{\sigma}\,(t,\lambda)\, u^{\sigma}\,(t,\mu)\,.
\end{aligned}
$$

Dividing both sides of above equality by $\lambda - \mu$ and taking limit as $\mu \to \lambda$, we have

$$
\lim_{\mu\to\lambda} \frac{\left\{ u\,(t,\mu)\, u^{\Delta}\,(t,\lambda) - u^{\Delta}\,(t,\mu)\, u\,(t,\lambda) \right\}^{\Delta}}{\lambda-\mu} = -\lim_{\mu\to\lambda} \left\{ u^{\sigma}\,(t,\lambda)\, u^{\sigma}\,(t,\mu) \right\}
$$

$$
\Rightarrow \left\{ u\,(t,\lambda)\,\frac{\partial}{\partial\lambda} u^{\Delta}\,(t,\lambda) - u^{\Delta}\,(t,\lambda)\,\frac{\partial}{\partial\lambda} u\,(t,\lambda) \right\}^{\Delta} = -\left(u^{\sigma}\,(t,\lambda) \right)^{2}.
$$

By taking $\Delta-$integral of the last equality from $\rho(a)$ to t, we get

$$\int\limits_{\rho(a)}^{t} \left\{ u(\tau,\lambda) \frac{\partial}{\partial\lambda} u^{\Delta}(\tau,\lambda) - u^{\Delta}(\tau,\lambda) \frac{\partial}{\partial\lambda} u(\tau,\lambda) \right\}^{\Delta} \Delta\tau = -\int\limits_{\rho(a)}^{t} \left(u^{\sigma}(\tau,\lambda)\right)^2 \Delta\tau.$$

Since $u(\rho(a),\lambda) = 1$ and $u^{\Delta}(\rho(a),\lambda) = 0$, it yields $\frac{\partial}{\partial\lambda} u(\rho(a),\lambda) = 0$ and $\frac{\partial}{\partial\lambda} u^{\Delta}(\rho(a),\lambda) = 0$. Finally, after some computations, we obtain

$$u(t,\lambda) \frac{\partial}{\partial\lambda} u^{\Delta}(t,\lambda) - u^{\Delta}(t,\lambda) \frac{\partial}{\partial\lambda} u(t,\lambda) = -\int\limits_{\rho(a)}^{t} \left(u^{\sigma}(\tau,\lambda)\right)^2 \Delta\tau.$$

So, the proof is completed.

Theorem 5.5. The eigenfunctions $u_1(t,\lambda_1)$ and $u_2(t,\lambda_2)$ of the problem (5.1)-(5.3) corresponding to distinct eigenvalues λ_1 and λ_2, are orthogonal , i.e

$$\int\limits_{\rho(a)}^{b} u_1^{\sigma}(t,\lambda_1) u_2^{\sigma}(t,\lambda_2) \Delta t = 0.$$

Proof: Let us use following equality to prove orthogonality.

$$\left\{ u_1^{\Delta}(t,\lambda_1) u_2(t,\lambda_2) - u_2^{\Delta}(t,\lambda_2) u_1(t,\lambda_1) \right\}^{\Delta} = (\lambda_2 - \lambda_1)\, u_1^{\sigma}(t,\lambda_1) u_2^{\sigma}(t,\lambda_2).$$

Taking $\Delta-$integral of the last equality from $\rho(a)$ to b, we get

$$\int\limits_{\rho(a)}^{b} \left\{ u_1^{\Delta}(t,\lambda_1) u_2(t,\lambda_2) - u_2^{\Delta}(t,\lambda_2) u_1(t,\lambda_1) \right\}^{\Delta} \Delta t = (\lambda_2 - \lambda_1) \int\limits_{\rho(a)}^{b} u_1^{\sigma}(t,\lambda_1) u_2^{\sigma}(t,\lambda_2) \Delta t$$

and

$$\int\limits_{\rho(a)}^{b} u_1^{\sigma}(t,\lambda_1) u_2^{\sigma}(t,\lambda_2) \Delta t = 0,$$

by using the conditions (5.2) and (5.3), where $\lambda_1 \neq \lambda_2$. Then, it shows that the eigenfunctions $u_1(t,\lambda_1)$ and $u_2(t,\lambda_2)$ corresponding to distinct eigenvalues are always orthogonal.

5.2 Main Results on Bessel equation on time scales

In this section, we obtain asymptotic estimates for the eigenfunction $u(t,\mu)$ (and its $\Delta-$ derivative) of the problem (5.1)-(5.3) on $\mathbb{T}$ whose all points are right dense.

Theorem 5.6. The eigenfunction $u(t,\mu)$ and its $\Delta-$ derivative for the problem (5.1)-(5.3) have the following asymptotic estimates;

$$u(t,\mu) = \cos_\mu(t,\rho(a)) - \frac{1}{\mu}\int\limits_{\rho(a)}^{t}\left[w_0(s) + \frac{l(l+1)}{s^2}\right]u(s,\mu)\sin_\mu(t,s)\Delta s,$$

and

$$u^\Delta(t,\mu) = -\mu\sin_\mu(t,\rho(a)) - \int\limits_{\rho(a)}^{t}\left[w_0(s) + \frac{l(l+1)}{s^2}\right]u(s,\mu)\cos_\mu(t,s)\Delta s,$$

respectively, where $t \in [\rho(a),b] \cap \mathbb{T}$ and $\mu^2 = \lambda$.

Proof: Let us consider the equation (5.1). First of all, we need to find the homogeneous solution of this equation. By homogeneous form of (5.1), one can easily get

$$u^{\Delta\Delta} + \mu^2 u = 0.$$

The homogeneous solution of above equation is

$$u(t,\mu) = c_1\cos_\mu(t,t_0) + c_2\sin_\mu(t,t_0), \tag{5.2.1}$$

where $t_0 \in [\rho(a),b] \cap \mathbb{T}$. Now, we are ready to apply generalized variation of parameters method to find particular solution of the equation (5.1). By (5.1), we have

$$\begin{aligned} c_1^\Delta\cos_\mu(t,t_0) + c_2^\Delta\sin_\mu(t,t_0) &= 0, \\ -\mu c_1^\Delta\sin_\mu(t,t_0) + \mu c_2^\Delta\cos_\mu(t,t_0) &= w(t)u. \end{aligned}$$

By multiplying these equations by $\mu\cos_\mu(t,t_0)$, $\mu\sin_\mu(t,t_0)$, and $-\sin_\mu(t,t_0)$, $\cos_\mu(t,t_0)$ respectively, we get

$$\begin{aligned} c_1^\Delta &= -\frac{1}{\mu}w(t)u(t,\mu)\sin_\mu(t,t_0), \\ c_2^\Delta &= \frac{1}{\mu}w(t)u(t,\mu)\cos_\mu(t,t_0). \end{aligned}$$

Integrating these equalities and considering (5.2.1), we obtain

$$\begin{aligned} u(t,\mu) &= \cos_\mu(t,t_0)\int\limits_{\rho(a)}^{t}\left(-\frac{1}{\mu}w(s)u(s,\mu)\sin_\mu(s,t_0)\right)\Delta s \\ &\quad + \sin_\mu(t,t_0)\int\limits_{\rho(a)}^{t}\left(\frac{1}{\mu}w(s)u(s,\mu)\cos_\mu(s,t_0)\right)\Delta s \end{aligned} \tag{5.2.2}$$

$$= \frac{1}{\mu}\int_{\rho(a)}^{t} \left(-\cos_\mu(t,t_0)\sin_\mu(s,t_0) + \sin_\mu(t,t_0)\cos_\mu(s,t_0)\right) w(s)u(s,\mu)\Delta s$$

$$= \frac{1}{\mu}\int_{\rho(a)}^{t} \sin_\mu(t,s)w(s)u(s,\mu)\Delta s.$$

By equation (5.1), we can write that

$$\begin{aligned} \frac{1}{\mu}\int_{\rho(a)}^{t} \sin_\mu(t,s)w(s)u(s,\mu)\Delta s &= \frac{1}{\mu}\int_{\rho(a)}^{t} \sin_\mu(t,s)\left(u^{\Delta\Delta}(s,\mu)+\mu^2 u(s,\mu)\right)\Delta s \\ &= \frac{1}{\mu}\int_{\rho(a)}^{t} \sin_\mu(t,s)u^{\Delta\Delta}(s,\mu)\Delta s + \mu\int_{\rho(a)}^{t} \sin_\mu(t,s)u(s,\mu)\Delta s. \end{aligned} \tag{5.2.3}$$

After using integration by parts twice the first integral in (5.2.3), we get

$$\begin{aligned} \frac{1}{\mu}\int_{\rho(a)}^{t} \sin_\mu(t,s)u^{\Delta\Delta}(s,\mu)\Delta s &= \frac{1}{\mu}\left(\sin_\mu(t,s)u^{\Delta}(s,\mu)\right)\Big|_{\rho(a)}^{t} - \frac{1}{\mu}\int_{\rho(a)}^{t} \mu\cos_\mu(t,s)u^{\Delta}(s,\mu)\Delta s \\ &= \frac{1}{\mu}\sin_\mu(t,t)u^{\Delta}(t,\mu) - \frac{1}{\mu}\sin_\mu(t,\rho(a))u^{\Delta}(\rho(a),\mu) \\ &\quad - \left(\cos_\mu(t,s)u(s,\mu)\right)\Big|_{\rho(a)}^{t} + \int_{\rho(a)}^{t} -\mu\sin_\mu(t,s)u(s,\mu)\Delta s \\ &= -\frac{1}{\mu}\sin_\mu(t,\rho(a))u^{\Delta}(\rho(a),\mu) - u(t,\mu) \\ &\quad + \cos_\mu(t,\rho(a))u(\rho(a),\mu) - \mu\int_{\rho(a)}^{t} \sin_\mu(t,s)u(s,\mu)\Delta s, \end{aligned}$$

and by using conditions (5.2) and (5.3), we get

$$\frac{1}{\mu}\int_{\rho(a)}^{t} \sin_\mu(t,s)u^{\Delta\Delta}(s,\mu)\Delta s = -u(t,\mu) + \cos_\mu(t,\rho(a)) - \mu\int_{\rho(a)}^{t} \sin_\mu(t,s)u(s,\mu)\Delta s. \tag{5.2.4}$$

Therefore, by considering (5.2.3) and (5.2.4), we have

$$u(t,\mu) = \cos_\mu\left(t,\rho(a)\right) - \frac{1}{\mu}\int_{\rho(a)}^{t} w(s)u(s,\mu)\sin_\mu(t,s)\Delta s. \tag{5.2.5}$$

Differentiating (5.2.5) with respect to t, we obtain

$$u^{\Delta}(t,\mu) = -\mu\sin_\mu\left(t,\rho(a)\right) - \frac{1}{\mu}\sin_\mu\left(\sigma(t),t\right)w(t)u(t,\mu) - \frac{1}{\mu}\int_{\rho(a)}^{t} w(s)u(s,\mu)\mu\cos_\mu\left(t,s\right)\Delta s,$$

or since $\sin_\mu(\sigma(t), t) = \sin_\mu(t, t) = 0$, we have

$$u^\Delta(t, \mu) = -\mu \sin_\mu(t, \rho(a)) - \int_{\rho(a)}^{t} w(s) u(s, \mu) \cos_\mu(t, s) \Delta s. \tag{5.2.6}$$

This completes the proof.

Bessel eigenvalue problems occur in many fields of science and have many applications in mathematical physics. We have focused on to get some asymptotic estimates of eigenfunctions for Bessel eigenvalue problem on $\mathbb{T}$. We have obtained some good results to fill the gaps in this area. We hope that this results will give a lead to the mathematicians. However, our main goal is to define inverse problem for different type operators on $\mathbb{T}$. We see this study as a milestones to consider inverse spectral theory on time scale.

6 CONCLUSION

There are many studies about the application of time scales to different areas. But, there is a limited number of studies about spectral theory on time scales. In literature, we can see some important studies of Guseinov, Bohner and Agarwal. In this thesis, we have studied on spectral theory of some differential operators on an arbitrary time scale to fill this gap. Spectral theory on time scales is new and original subject. We are sure that our results will make a great contribution to spectral theory. As far as our examinations, many topics of spectral theory are not defined on time scales. Hence, we tried to define Dirac system, impulsive diffusion equation and bessel equation on time scales. Later, we have obtained some results about the basic spectral properties of these differential operators on time scales. Furthermore, we have constructed the asymptotic estimates for the eigenfunctions of the problems which contains these differential operators on time scales. According to our opinion, it will be an important milestone in spectral theory. Because, one can use these results to get asymptotic estimates of the eigenvalues for these type operators on time scale. Then, it can be possible to give some results about Ambarzumyan type theorems on time scales. As known, Ambarzumyan's theorem is the first study about inverse problem. Therefore, the first results about inverse problems in spectral theory will be obtained on a time scale. If everything goes right, inverse nodal problem can be defined on a time scale in next times. This thesis will allow many mathematicians to examine the topic in discrete situations. We are sure that many young mathematicians will build their academic life on these results. On the other hand, we know that there are many application of inverse spectral theory in real life. All of these applications can be reconsidered on time scales. If mathematicians apply inverse spectral theory and their applications to the time scales, many problems that humanity can not solve can be solved. However, the task of mathematicians is to work constantly, without hesitation, and to sign work that will be useful. Even the smallest contribution, we can make to this great discipline to be happy. For this reason, we intend to continue our studies with great determination.

References

[1] **Hilger, S.,** 1988. Ein Masskettenkalkül mit Anwendung auf Zentrumsmannigfaltigkeiten, *Ph.D. Thesis*, Universität Würzburg, Germany.

[2] **Hilger, S.,** 1990. Analysis on measure chains-A unified approach to continuous and discrete calculus, *Results Math.,* **18,** 18-56.

[3] Aulbach, B. and **Hilger, S.,** 1990. A unified approach to continuous and discrete dynamics, *Qualitative theory of differential equations*, Szeged, 1988; Colloquia Mathematica Societatis János Bolyai, North-Holland, Amsterdam, **53,** 37–56.

[4] **Bohner, M.,** 2004. Calculus of variations on time scales, *Dynamic Systems and Applications*, **13**(3-4), 339-349.

[5] **Hilscher, R. and Zeidan, V.,** 2004. Calculus of variations on time scales: weak local piecewise C^1_{rd} solutions with variable endpoints, *J. Math. Anal. Appl.*, **289,** 143-166.

[6] **Atici, F.M. and Guseinov, G.S.,** 2002. On Green's functions and positive solutions for boundary value problems on time scales, *J. Comput. Appl. Math.,* **141,** 75–99.

[7] **Karpuz, B.,** 2016. Nonoscillation and oscillation of second order linear dynamic equations via the sequence of functions technique, *J. Fixed Point Theory Appl.,* 1-15 Doi: 10.1007/511784.016.0334.8.

[8] **Caputo, M.C.,** 2009. Time scales: from nabla calculus to delta calculus and vice versa via duality, *arXiv.org > math > arXiv:0910.0085.*

[9] **Bohner, M. and Peterson, A.,** 2001. *Dynamic equations on time scales: an introduction with applications*, Birkhäuser Boston Inc, Boston (MA).

[10] **Bohner, M. and Peterson, A.,** 2003. *Advances in dynamic equations on time scales*, Birkhäuser Boston Inc., Boston, MA.

[11] **Agarwal, R.P., Ertem, T. and Zafer, A.,** 2015. Asymptotic integration of second-order nonlinear delay differential equations, *Appl. Math. Lett.*, **48,** 128-134.

[12] **Chyan, C.J., Davis, J.M. and Henderson, J.,** 1998. Eigenvalue comparisons for differential equations on a measure chain, Electron. *J. Diff. Equ.,* **35,** 1-7.

[13] **Agarwal, R.P., Bohner, M. and Wong, J.Y.,** 1999. Sturm–Liouville eigenvalue problems on time scales, *Appl. Math. Comput.,* **99,** 153–166.

[14] **Agarwal, R.P., Bohner, M. and O'Regan, D.,** 2002. Time scale boundary value problems on infinite intervals, *J. Comput. Appl. Math.,* **141,** 27–34.

[15] **Guseinov, G.Sh.,** 2007. Eigenfunction expansions for a Sturm–Liouville problem on time scales, *Int. J. Diff. Equ.,* **2**, 93–104.

[16] **Huseynov, A. and Bairamov, E.,** 2009. On expansions in eigenfunctions for second order dynamic equations on time scales, *Nonlinear Dyn. Syst. Theory.*, **9,** 77-88.

[17] **Zhang, Y. and Ma, L.,** 2010. Solvability of Sturm-Liouville problems on time scales at resonance, *J. Comput. Appl. Math.,* **233,** 1785-1797.

[18] **Erbe, L., Mert, R. and Peterson, A.,** 2012. Spectral parameter power series for Sturm–Liouville equations on time scales, *Appl. Math. Comput.*, **218,** 7671-7678.

[19] **Tuna, H.,** 2014. Completeness of the rootvectors of a dissipative Sturm–Liouville operators on time scales, *Appl. Math. Comput.,* **228,** 108-115.

[20] **Gulsen, T. and Yilmaz, E.,** 2016. Spectral theory of Dirac system on time scales, *Appl. Anal.,* 1-11, Doi: 10.1080/00036811.2016.1236923.

[21] **Guseinov, G.Sh.,** 2005. Self-adjoint boundary value problems on time scales and symmetric Green's functions, *Turkish J. Math.,* **29,** 365–380.

[22] **Guseinov, G.Sh.,** 2007. Eigenfunction expansions for a Sturm–Liouville problem on time scales, *International Journal of Difference Equations*, **2**(1), 93–104.

[23] **E. S. Panakhov, E.S., Yilmaz, E. and Koyunbakan, H.,** 2010. Inverse nodal problem for Dirac operator, *World Applied Sciences Journal*, **11**(8), 906-911.

[24] **Moses, H.E.,** 1956. Calculation of the scattering potential from reflection coefficients, *Physical Review*, **102**(2), 557-559.

[25] **Prats, F. and Toll, J.S.,** 1959. Construction of the Dirac equation central potential from phase shifts and bound states, *Physical Review*, **113**(1), 363-370.

[26] **Verde, M.,** 1959. The inversion problem in wave mechanics and dispersion relations, *Nuclear Physics*, **9**(2), 255-266.

[27] **Gasymov, M.G. and Levitan, B.M.,** 1966. The inverse problem for a Dirac System, *Doklady Akademi NAauk SSSR*, **167**, 967-970.

[28] **Panakhov, E.S.,** 1985. The defining of Dirac system in two incompletely set collection of eigenvalues, *Doklady Akademi AzSSR*, **5**, 8-12.

[29] **Gasymov M.G. and Dzhabiev, T.T.,** 1975. Determination of a system of Dirac differential equations using two spectra, *Transactions of the summer school on spectral Theory of Operators*, 46-71.

[30] **Levitan, B.M. and Sargsjan, I.S.,** 1991. *Sturm-Liouville and Dirac operators*, Nauka, Moscow.

[31] **Joa, I. and Minkin, A.,** 1997. Eigenfunction estimate for a Dirac operator, *Acta Mathematica Hungarica*, **76**(4), 337-349.

[32] **Watson, B.A.,** 1999. Inverse spectral problems for weighted Dirac systems, *Inverse Problems*, **15**(3), 793-805.

[33] **Annaby M.H.and Tharwat, M.M.,** 2011. On sampling and Dirac systems with eigenparameter in the boundary conditions, *Journal of Applied Mathematics and Computing*, **36**(1-2), 291-317.

[34] **Amirov, R.K., Keskin, B.and Özkan, A.S.,** 2009. Direct and inverse problems for the Dirac operator with a spectral parameter linearly contained in a boundary condition, *Ukranian Mathematical Journal*, **61**(9), 1365-1379.

[35] **Yang, C.F. and Huang, Z.Y.,** 2010. Reconstruction of the Dirac operator from nodal data, *Integral Equations and Operator Theory*, **66**(4), 539-551.

[36] **Wei, Z., Guo, Y. and Wei, G.,** 2015. Incomplete inverse spectral and nodal problems for Dirac operator, *Advances in Difference Equations*, **1**, 1-12.

[37] **Yang, C.F.,** 2016. Inverse problems for Dirac equations polynomially depending on the spectral parameter, *Applicable Analysis*, **95**(6).

[38] **Bairamov, E., Aygar Y. and Olgun, M.,** 2010. Jost solution and the spectrum of the discrete Dirac systems, *Bound. Value Probl.*, Art. ID 306571, 11 pp.

[39] **Kerimov, N.B.,** 2002. A boundary value problem for Dirac system with a spectral parameter in the boundary conditions, *Differential Equations*, **38**(2), 164-174.

[40] **Yang, C.F. and Pivovarchik, V.N.,** 2011. Inverse nodal problem for Dirac system with spectral parameter in boundary conditions, *Complex Analysis and Operator Theory*, **7**(4), 1211-1230.

[41] **Guo, Y. and Wei, G.,** 2015. Inverse nodal problem for Dirac equations with boundary conditions polynomially dependent on the spectral parameter, *Results in Mathematics*, **67**(1-2), 95-110.

[42] **Cakmak, Y. and Isık, S.,** 2016. Half Inverse problem for the impulsive diffusion operator with discontinuous coefficient, *Filomat*, **30**(1), 157-168.

[43] **Jaulent, M. and Jean, C.,** 1972.The inverses-wave scattering problem for a class of potentials depending on energy, *Communications in Mathematical Physics*, **28**(3), 177-220.

[44] **Wazwaz, A.**, 2002. Partial differential equations: Methods and applications, Balkema Publishers, Leiden.

[45] **Maksudov, F.G. and Guseĭnov, M.M.,** 1979. A quadratic pencil of operators in the presence of a continuous spectrum. (Russian) Akad. Nauk Azerbaĭdzhan. *SSR Dokl.,* **35**(1), 9–13.

[46] **Gasymov, M.G. and Guseinov, G.Sh.,** 1981. Determination of a diffusion operator from the spectral data, *Doklady Akademii Nauk Azerbaijan SSSR*, **37**(2), 19-23.

[47] **Fragela, A.,** 1984. Quadratic pencils of differential operators with integral boundary conditions, (Russian) *Differential equations and their applications* (Russian), 50–52, Moskov. Gos. Univ., Moscow.

[48] **Maksudov, F.G., Guseĭnov, G.Sh.,** 1986. On the solution of the inverse scattering problem for a quadratic pencil of one-dimensional Schrödinger operators on the whole axis, (Russian) *Dokl. Akad. Nauk SSSR,* **289**(1), 42–46.

[49] **Bairamov, E., Çakar, Ö. and Çelebi, A.O.,** 1997. Quadratic pencil of Schrödinger operators with spectral singularities: discrete spectrum and principal functions, *Journal of Mathematical Analysis and Applications*, **216**(1), 303-320.

[50] **Koyunbakan, H.,** 2006. A new inverse problem for the diffusion operator, *Applied Mathematics Letters*, **19**(10), 995-999.

[51] **Koyunbakan, H. and Panakhov, E.S.,** 2007. A uniqueness theorem for inverse nodal problem, *Inverse Problems in Science and Engineering*, **5**(6), 517-524.

[52] **Koyunbakan, H.,** 2008. Reconstruction of potential function for diffusion operator, *Numerical Functional Analysis and Optimization,* **29**(7-8), 1-10.

[53] **Koyunbakan, H. and Yilmaz, E.,** 2008. Reconstruction of the potential function and its derivatives for the diffusion operator, *Verlag der Zeitschrift für Naturforchung A*, **63**(3-4), 127-130.

[54] **Yang, C.F.,** 2010. Reconstruction of the diffusion operator from nodal data, *Verlag der Zeitschrift für Naturforchung A*, **65**(1-2), 100-106.

[55] **Yang, C.F. and Zettl, A.,** 2012. Half inverse problems for quadratic pencils of Sturm-Liouville operators, *Taiwanese Journal of Mathematics*, **16**(5), 1829-1846.

[56] **Hryniv, R. and Pronska, N.,** 2012. Inverse spectral problems for energy-dependent Sturm-Liouville equations, *Inverse Problems*, **28**(8), 085008.

[57] **Buterin, S.A. and Shieh, C.T.,** 2009. Inverse nodal problem for differential pencils, *Applied Mathematics Letters*, **22**, 1240-1247.

[58] **Yakhshimuratov, A.B., Allaberganov, O.R.,** 2006. The inverse problem for a quadratic pencil of Sturm-Liouville operators with a periodic potential on a half-axis, (Russian) *Uzbek. Mat. Zh.,* **3**, 96–107.

[59] **Guseinov, I.M. and Nabiev, I.M.,** 2007. The inverse spectral problem for pencils of differential operators, *Sbornik Mathematics,* **198**(11), 1579-1598.

[60] **Guseĭnov, I.M. and Nabiev, I.M.,** 2000. A class of inverse problems for a quadratic pencil of Sturm-Liouville operators, (Russian) *Differ. Uravn.,* **36**(3) (2000), 418–420, 432; translation in Differ. Equ. **36**(3), 471–473.

[61] **Koyunbakan, H.,** 2011. Inverse problem for a quadratic pencil of Sturm-Liouville operator, *J. Math. Anal. Appl.,* **378**(2), 549–554.

[62] **Wang, Y.P.,** 2012. The inverse problem for differential pencils with eigenparameter dependent boundary conditions from interior spectral data, *Applied Mathematics Letters*, **25**(7), 1061-1067.

[63] **Amirov, R.Kh. and Nabiev, A.A.,** 2013. Inverse problems for the quadratic pencil of the Sturm-Liouville equations with impulse, *Abstr. Appl. Anal.,* **2013**, Art. ID 361989, 10 pp.

[64] **Sharma, L.K., Luhanga, P.V. and Chimidza, S.,** 2011. Potentials for the Klein-Gordon and Dirac equations, *Chiang Mai Journal of Science,* **38**(4), 514-526.

[65] **Chadan, K., Colton, D., Paivarinta, L. and Rundell, W.,** 1997. An introduction to inverse scattering and inverse spectral problems, *SIAM*, Philadelphia, PA.

[66] **Orujov, Ashraf D.,** 2015. On the spectrum of the quadratic pencil of differential operators with periodic coefficients on the semi-axis, *Bound. Value Probl.,* 2015:117, 16 pp.

[67] **Watson, G.N.,** 1995. *A Treatise on the Theory of Bessel Functions*, reprint of the second ed., Cambridge University Press, Cambridge.

[68] **Albeverio, S., Hyrniv, R. and Mykytyuk, Ya.,** 2007. Inverse spectral problems for Bessel operators, *J. Diff. Op.,* **241,** 130-159.

[69] **Willson, R.W. and Peirce, B.O.,** 1897. Table of the first forty roots of the Bessel equation $J_0(x) = 0$ with the corresponding values of $J_1(x)$, *Bulletin American Math. Society*, **3,** 153-155.

[70] **Chessin, A.,** 1894. Note on the general solution of the Bessel's equation, *American J. Math.,* **16,** 186-187.

[71] **Stashevskaya, V.V.,** 1953. On inverse problem of spectral analysis for a class of differential equations, *Doklady Akademii Nauk SSSR.,* **93,** 409-411.

[72] **Pöschel, J. and Trubowitz, E.,** 1987. Inverse Spectral Theory, *Pure Appl. Math.,* **130,** Academic Press, Orlando, FL.

[73] **Guillot, J.C. and Ralston, J.V.,** 1988. Inverse spectral theory for a singular Sturm-Liouville operator on $[0, 1]$, *J. Diff. Equ.,* **76,** 353-373.

[74] **Serier, F.,** 2007. Inverse spectral problem for radial Schrödinger operator on [0, 1], *J. Diff. Equ.,* **235,** 101–126.

[75] **Carlson, R.,** 1993. Inverse spectral theory for some singular Sturm–Liouville problems, *J. Diff. Equ.,* **106,** 121–140.

[76] **Zhornitskaya, L.A. and Serov, V.S.,** 1994. Inverse eigenvalue problems for a singular Sturm–Liouville operator on [0, 1], *Inv. Probl.,* **10,** 975–987.

[77] **Carlson, R.A.,** 1997. A Borg-Levinson theorem for Bessel operators, *Pacific J. Math.,* **177,** 1-26.

[78] **Gasymov, M.G.,** 1965. Determination of a Sturm–Liouville equation with a singularity by two spectra, *Dokl. Akad. Nauk SSSR,* **161,** 274–276 (1965) (in Russian); Engl. transl.: *Soviet Math. Dokl.,* **6,** 396–399.

[79] **Andersson, L.E.,** 1988. Inverse eigenvalue problems with discontinuous coefficients, *Inv. Probl.,* **4,** 353–397.

[80] **Marchenko, V.A.,** 1986. *Sturm–Liouville Operators and Their Applications*, Naukova Dumka Publ., Kiev, 332 p. (1977) (in Russian); Engl. transl.: Birkhäuser Verlag, Basel.

[81] **Christ, C.S.,** 1993. An inverse problem for the Schrödinger equation with a radial potential, *J. Diff. Equ.,* **103,** 247–259.

[82] **Titchmarsh, E.C.,** 1962. *Eigenfunction expansions associated with second order differential equations: I,* Clarendon Press, Oxford.

[83] **Topsakal, N. and Amirov, R.,** 2010. Inverse problem for Sturm–Liouville operators with Coulomb potential which have discontinuity conditions inside an interval, *Math. Phys. Analy. Geo.,* **13,** 29-46.

[84] **Levitan, B.M.,** 1987. *Inverse Sturm-Liouville problems*, Netherland VNU Science Press.

[85] **Yurko, V.,** 2011. Inverse problems for Bessel type differential equations on noncompact graphs using spectral data, *Inv. Probl.,* **27,** 045002 (17pp).

[86] **Koyunbakan, H. and Panakhov, E.S.,** 2006. Solution of a discontinuous inverse nodal problem on a finite interval, *Math. Comput. Modelling.,* **44,** 204-209.

[87] **Yilmaz, E. and Koyunbakan, H.,** 2016. Some Ambarzumyan type theorems for Bessel operator on a finite interval, *Diff. Equ. Dyn.Syst.,* 1-7.

BACKGROUND

My name is Shaida Saber Mawlood Sian. I was born in Erbil, Iraq in 1982. I was graduated from college of science, department of mathematics at university of Salahaddin in Erbil, Iraq in the academic year of 2007-2008. I was admitted to graduate program at Firat University, Elazığ, Türkiye in 2015. In 2016, I have got my MSc in Mathematics at Firat University, Elazığ-Türkiye. Right now, I am working at a high school as a teacher in Twana English mixed secondary school. I can speak Arabic and English.