

REPUBLIC OF TURKEY
FIRAT UNIVERSITY
THE GRADUATE SCHOOL OF NATURAL AND APPLIED SCIENCE



**APPLICATIONS OF THE $\tan\left(\frac{F(\xi)}{2}\right)$ EXPANSION METHOD
TO THE NONLINEAR PARTIAL DIFFERENTIAL EQUATIONS**

RABAR MOHAMMED RASUL
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Master Thesis
Department: Mathematics
Supervisor: Prof. Dr. Hasan BULUT
April-2017

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ELAIZIG – 2017

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SUMMARY

This work is made up of the five sections.

In section one, we describe and show some basic definitions that are must be in this study.

In section two, we going to show the general structures of the $\tan\left(\frac{F(\xi)}{2}\right)$ -expansion method.

In section three, we apply the $\tan\left(\frac{F(\xi)}{2}\right)$ -expansion method to the potential KdV and Calogero–Bogoyavlenskii–Schiff equations. We obtain a new solution to these equations like rational function solutions, exponential function, hyperbolic function and trigonometric function. We carry out all the computations in this study with the Wolfram Mathematica 9.

In chapter four, we present the two- and three-dimensional graphics of some obtained solutions plotted by using the same program in the Wolfram Mathematica 9.

In chapter five, we give a comprehensive conclusion to this study.

ÖZET

$Tan\left(\frac{F(\xi)}{2}\right)$ AÇILIM METODUNUN LINEER OLMAYAN KISMI DIFERANSİYEL DENKLEMLERE UYGULAMASI

Bu çalışma beş bölümden oluşmaktadır.

Birinci bölümde, bu tezde gerekli olan bazı temel tanımlar verildi.

İkinci bölümde; $tan\left(\frac{F(\xi)}{2}\right)$ açılım metodunun genel yapısı sunuldu.

Üçüncü bölümde; Calogero-Bogoyavlenskil-Schiff ve potensiyel KDV denkleminde $tan\left(\frac{F(\xi)}{2}\right)$ açılım metodunu uyguladık. Bu denklemlerin rasyonel, üstel, hiperbolik ve trigonometrik yeni çözümlerini elde ettik. Mathematica 9 programını kullanarak bu çalışmadaki cebirsel hesaplamaları yaptık.

Dördüncü bölümde ise, Mathematica 9 programını kullanarak elde ettiğimiz bu yeni çözümlerin iki ve üç boyutlu grafikleri çizildi.

Beşinci bölümde; bu çalışma hakkında kapsamlı bir sonuç verildi.

INTRODUCTION

Nonlinearity has an important role in applied sciences. Calculating traveling wave solutions of nonlinear equations is very significant in mathematical physics [1, 2]. Many analytical methods have been found in literature [3-11]. Besides these methods, there are many methods which reach to solution by using an auxiliary equation. Firstly, the nonlinear partial differential equations (NPDEs) are reduced to nonlinear ordinary differential equations (NODEs) by using these methods. Second, the obtained NODEs are solved with the help of the auxiliary equation. These methods can be listed as: generalized Jacobi elliptic function method [12], tanh function method [13], $\frac{G'}{G}$ -expansion method [14], generalized $\frac{G'}{G}$ -expansion method [15], $(\frac{G'}{G}, \frac{1}{G})$ -expansion method [16], extended $\frac{G'}{G}$ -expansion method [17], extended tanh function method [18], modified extended tanh method [19], improved tanh function method [20], Jacobi elliptic function method [21], extended Jacobi elliptic function method [22], Jacobi elliptic rational expansion method [23], Weierstrass Jacobi elliptic function expansion method [24].

However, in this study we implement the $\tan(F(\xi)/2)$ -expansion Method [25], for finding the exact solutions of the (2+1)-dimensional potential KdV equation [26] and the (2+1)-dimensional Calogero–Bogoyavlenskii–Shiff equation [27]. Many scientists have studied on $\tan(F(\xi)/2)$ -expansion method [28-30] in literature.

1. FUNDAMENTAL DEFINITIONS

1.1 Definitions [31]

1.1.1 Definition of Partial differential equations (PDE)

Partial differential equations (PDEs) are equations that can be used to describe some aspects in the field of nonlinear sciences such as physics, biology, chemistry etc. PDEs involve the dependent variable, and its partial derivatives. We can see that in the (ODEs), the dependent variable $v = v(x)$, depends just on single independent variable x different from the ODEs, the dependent variable in the PDEs, like $v = v(x, t)$ have to depend on more than one independent variables. Unless $v = v(x, t)$, after that the function v depends on the independent variable x and t . However, if $v = v(x, y, t)$, then the function v depends on the space variables x, y , and t .

The following PDEs can be given by

$$v_t = cv_{xx}, \quad (1.1)$$

$$u_t = cv_{xx} + cv_{yy}, \quad (1.2)$$

$$v_t = cv_{xx} + cv_{yy} + cv_{zz}. \quad (1.3)$$

That describes the heat flow in one dimensional space, two dimensional space, and three dimensional space appropriately. The dependent variable In Eq. (1.1), $v = v(x, t)$ depends on the location x and t . However, in Eq. (1.2), $v = v(x, y, t)$ depends on three independent variables, the space variables x, y and t . In Eq. (1.3), the dependent variable $v = v(x, y, z, t)$ depends on four independent variables, the space variables x, y, z and t .

Some other instances of PDE can be given

$$v_{tt} = k^2 v_{xx}, \quad (1.4)$$

$$v_{tt} = k^2 v_{xx} + k^2 v_{yy}, \quad (1.5)$$

$$v_{tt} = k^2 v_{xx} + k^2 v_{yy} + k^2 v_{zz}. \quad (1.6)$$

That describes the wave propagation in one dimensional space, two dimensional spaces, and three dimensional spaces appropriately. moreover, the unidentified functions in (1.4), (1.5), and (1.6) are defined by $v = v(x, t)$, $v = v(x, y, t)$ and $v = v(x, y, z, t)$ respectively.

The popularly known Laplace equation can be given by

$$v_{yy} + v_{xx} = 0 , \quad (1.7)$$

$$v_{yy} + v_{xx} + v_{zz} = 0 . \quad (1.8)$$

Where the function v is not depends on variable t . As will be shown, the Laplace's equation in polar coordinates can be given by

$$v_y + \frac{1}{r} v_r + \frac{1}{r^2} v_{\theta\theta} = 0, \quad (1.9)$$

Where $v = v(r, \theta)$.

Furthermore, the Burgers equation and the KdV equation can be given by

$$v_t + 2vv_x - kv_{xx} = 0, \quad (1.10)$$

$$v_t + vv_x - \alpha v_{xxx} = 0 , \quad (1.11)$$

appropriately, where the function v depends on t and x .

1.1.2 Order of a PDE

In PDE the order of the highest partial derivative that be seen in the equation is the order of a PDE. For instance, see the equations below, the eq.(1.12) is the first order, the eq.(1.13) is the second order, and the eq.(1.14) the third order.

$$v_y - kv_t = 0 , \quad (1.12)$$

$$v_{xx} - kv_t = 0, \quad (1.13)$$

$$v_{xt} - vv_{xxx} = 0, \quad (1.14)$$

Where k is constant.

1.1.3 Linear and Nonlinear PDEs

The PDEs are categorized into two kind linear equations and nonlinear equations. A PDE is named linear if:

(a) The power of dependent variable and every partial derivative include in the equation is one.

(b) The coefficients of the dependent variable and the coefficients of every partial derivative are constants or independent variables. Furthermore, if every of these conditions are not satisfied, the equation is named nonlinear equation, for instance:

The Eq. (1.15) and (1.16) are linear and Eq. (1.17) and (1.18) are nonlinear.

$$xv_{xx} + v_{yy} = 0. \quad (1.15)$$

$$v_{rr} + \frac{1}{r}v_r + \frac{1}{r^2}v_{\theta\theta} = 0. \quad (1.16)$$

$$vv_t + xv_x = 3. \quad (1.17)$$

$$kv_x + \sqrt{v} = x. \quad (1.18)$$

Where k constant

1.1.4 Some Linear PDEs

Linear PDEs stands up in many fields of scientific applications, like the wave equation and the diffusion equation. In what follows, we show several of the popularly known models that are of significant concern:

1. The heat equation in one dimensional space can be given by following equation.

$$v_t = \beta v_{xx}, \quad (1.19)$$

where β is a constant.

2. The wave equation in one dimensional space can be given by following equation.

$$v_{tt} = b^2 v_{xx}, \quad (1.20)$$

where b is a constant.

3. The Laplace equation can be given by following equation.

$$v_{tt} + v_{xx} = 0. \quad (1.21)$$

4. The Klein-Gordon equation can be given by following equation.

$$\nabla^2 v - \frac{1}{k^2} v_{tt} = \lambda^2 v, \quad (1.22)$$

where λ and k are constants.

5. The Linear Schrodinger's equation can be given by following equation.

$$iv_t + v_{xx} = 0, i = \sqrt{-1}. \quad (1.23)$$

6. The Telegraph equation can be given by following equation.

$$v_{xx} = cv_{tt} + kv_{tt} + dv, \quad (1.24)$$

where (k, c and d) are constants.

1.1.5 Some Nonlinear PDEs

It was stated earlier that PDEs used to describe different aspects in the fields of engineering science and mathematical physics, including plasma physics, nonlinear fiber optics, fluid dynamics, nonlinear wave propagation and quantum field theory. In below examples we write some well-known of nonlinear models that are more important:

1. The Advection equation can be given by following example.

$$v_t + vv_x = g(x, t). \quad (1.25)$$

2. The Burgers equation can be given by following equation.

$$v_t + vv_x = \alpha v_{xx}. \quad (1.26)$$

3. The Korteweg de-Vries (KdV) equation can be given by following equation.

$$v_t + \alpha vv_x + \alpha v_{xxx} = 0. \quad (1.27)$$

4. The modified KdV equation (mKdV) can be given by following equation.

$$v_t + 6v^2v_x + v_{xxx} = 0. \quad (1.28)$$

5. The Boussinesq equation can be given by following equation.

$$v_{tt} - v_{xx} + 3(v^2)_{xx} - v_{xxx} = 0. \quad (1.29)$$

6. The sine-Gordon equation can be given by following equation.

$$v_{tt} - v_{xx} = \gamma \sin(v). \quad (1.30)$$

7. The sinh-Gordon equation can be given by following equation.

$$v_{tt} - v_{xx} = \gamma \sinh(v). \quad (1.31)$$

8. The Liouville equation can be given by following equation.

$$v_{tt} - v_{xx} = e^{\mp v}. \quad (1.32)$$

9. The Fisher equation can be given by following equation.

$$v_t = Av_{xx} + v(1 - v). \quad (1.33)$$

10. The Kadomtsev-Petviashvili (KP) equation can be given by following equation.

$$(v_t + \beta vv_x + \alpha v_{xxx})_x + v_{xx} = 0. \quad (1.34)$$

11. The K(n,n) equation can be given by following equation.

$$v_t + \gamma(v^n)_x + \alpha(v^n)_{xx} = 0, n > 1. \quad (1.35)$$

12. The Nonlinear Schrodinger (NLS) equation can be given by following equation.

$$iv_t + v_{xx} + \mu|v|^2 = 0. \quad (1.36)$$

13. The Camassa-Holm(CH) equation can be given by following equation.

$$v_t - v_{xxt} + \beta v_x + 3vv_x = 2v_xv + vv_{xxx}. \quad (1.37)$$

14. The Degasperis-Procesi (DP) equation can be given by following equation.

$$v_t - v_{xxt} + \beta v_x + 4vv_x = 3v_x v_{xx} + vv_{xxx}. \quad (1.38)$$

1.1.6 Homogeneous and Non-homogeneous PDEs

The equations of PDEs are also categorized into homogeneous and non-homogeneous. A PDEs of every order are termed as homogeneous equation if any term of the PDE contains the dependent variable v or one of its derivatives, on the other hand, it is termed as non-homogeneous equation. We can see this form in the following examples.

$$v_t = 4v_{xx}, \quad (1.39)$$

$$v_{xx} + v_{yy} = 0, \quad (1.40)$$

$$v_t = v_{xx} + x, \quad (1.41)$$

$$v_x + v_y = u + 4. \quad (1.42)$$

1.1.7 Solution of a PDE

A solution is a function that satisfies the equation of a PDE under consideration and satisfies the given conditions as well. On the other hand, the function can satisfy the equation, the left side and also the right side of the PDE must be equal when we substituting the outcomes of solution.

Remarks:

The remark below can be used in explaining the concept of a solution of PDE.

1. For a linear homogeneous ODE, it is popularly known that if $v_1, v_2, v_3, \dots, v_n$ are solutions of the equation, then a linear combination of $v_1, v_2, v_3, \dots$ shown by:

$$v = c_1 v_1 + c_2 v_2 + c_3 v_3 + \dots + v_n v_n, \quad (1.43)$$

is also a solution. The concept combines two or more of these solutions are named the superstition principle.

It is also worth nothing that the superstition principle tasks effectively for linear homogeneous equation in the domain of PDEs shown.

2. For a linear ODE, the general solution depends mainly on arbitrary constants. Different from ODE, in linear PDE, the general solution depends on arbitrary function. This can easily be scrutinized by considering the following PDE.

$$v_x + v_y = 0, \quad (1.44)$$

with its solution shown as

$$v = g(x - y), \quad (1.45)$$

where $g(x - y)$ is an arbitrary differentiable function. This shows that the solution of (1.42) can be every of functions below:

$$\begin{aligned} v &= x - 1, \\ v &= e^{x-y}, \\ v &= \sinh(x - y), \\ v &= \ln(x - y). \end{aligned} \quad (1.46)$$

And every function of the form $g(x - y)$ furthermore, the general solution of a PDE is of bit use. In reality, a particular solution that can satisfy prescribed conditions is always used.

1.2 Second-order PDE

A linear PDE that depends in two independent variables y and x is called second order, the general form can be given as a following equation:

$$av_{xx} + bv_{xy} + cv_{yy} + dv_x + fv_y + kv = g, \quad (1.47)$$

where a, b, c, d, f, k and g are constants or functions of the variables y and x .

A second order PDE (1.47) is usually categorized into three groups of equations:

1. Parabolic. Parabolic equation is an equation that satisfies the property

$$D^2 - 4AB = 0. \quad (1.48)$$

Examples of parabolic equations are heat flow and diffusion processes equations.

The heat transfer equation

$$v_t = Cv_{xx}. \quad (1.49)$$

2. Hyperbolic. Hyperbolic equation is an equation that satisfies the property

$$D^2 - 4AB > 0. \quad (1.50)$$

Examples of hyperbolic equations are wave propagation equations. The wave equation

$$v_{tt} = k^2 v_{xx}. \quad (1.51)$$

3. Elliptic. The equation that satisfies the property is Elliptic equation

$$D^2 - 4AB < 0. \quad (1.52)$$

Schrodinger equation and Laplace's equation are instances of elliptic equations.

The Laplace equation in a 2-dimentional space

$$v_{yy} + v_{xx} = 0. \quad (1.53)$$

Laplace's equation named the potential equation because $v(x,y)$ defines the potential function.



2. THE $\tan\left(\frac{F(\xi)}{2}\right)$ -EXPANSION METHOD

In section two, we have to give some descriptions about $\tan\left(\frac{F(\xi)}{2}\right)$ -expansion method[25] and how it works in finding new travelling wave solution to a given NPDEs. Consider the following NPDE;

$$Q(u, u_t, u_x, u_{xx}, \dots) = 0, \quad (2.1)$$

we transform Eq. (2.1) into nonlinear (ODE) by using the follow travelling wave transformation:

$$u(x, t) = u(\xi), \quad \xi = x - kt,$$

where k is an arbitrary constant. After the transformation, we obtain the following NODE in $u(\xi)$.

$$Q'(u', u'', u''', \dots) = 0. \quad (2.2)$$

Then, the solution of the Eq. (2.2) we can get:

$$u(\xi) = \sum_{i=0}^n A_i \left[p + \tan\left(\frac{F(\xi)}{2}\right) \right]^i + \sum_{i=1}^n B_i \left[p + \tan\left(\frac{F(\xi)}{2}\right) \right]^{-i}, \quad A_n \neq 0, B_n \neq 0, \quad (2.3)$$

n is a positive integer number that can be decided by balancing the highest nonlinear term with the highest order derivative in Eq. (2.2), the formula for balancing technique of NODE by considering the highest derivative $\frac{d^r u}{d\xi^r}$ in the NODE and the highest power nonlinear term $u^p \left(\frac{d^q u}{d\xi^q}\right)^s$, then we can find n by the following,

$$n + r = n.p + s(n + q). \quad (2.4)$$

The coefficients $A_i (0 \leq i \leq n), B_i (1 \leq i \leq n)$ are constant to be decided and $F = F(\xi)$ satisfies the following first order nonlinear ODE:

$$F'(\xi) = a \sin(F(\xi)) + b \cos(F(\xi)) + c. \quad (2.5)$$

Substituting Eq. (2.3) into Eq. (2.2) yields a set of algebraic equations for $\tan\left(\frac{F(\xi)}{2}\right)^i, \cot\left(\frac{F(\xi)}{2}\right)^i$, then, all coefficients of $\tan\left(\frac{F(\xi)}{2}\right)^i, \cot\left(\frac{F(\xi)}{2}\right)^i$ have to vanish. After this separated algebraic equations, we can find $p, k, B_1, A_0, A_1, \dots, B_n, A_n$ constants.

In this work, our purpose is to obtain the solutions of (2+1)-dimensional potential KdV equation by using $\tan\left(\frac{F(\xi)}{2}\right)$ -expansion method [25].



3. APPLICATION OF THE METHOD

In section three, we study two significant NPDEs by using the $\tan\left(\frac{F(\xi)}{2}\right)$ -expansion method to describe its efficiency.

Example1: Consider the (2+1)-dimensional potential KdV equation [26]:

$$u_{xt} + 2u_x u_{xx} + u_{xxxx} + u_{yy} = 0. \quad (3.1)$$

Using the travelling wave transformation $u(\xi) = u(x, y, t)$, $u(\xi) = x + \alpha y - kt$ on Eq. (3.1) we have the following:

$$-ku'' + 2u'u'' + u^{(4)} + \alpha^2 u'' = 0. \quad (3.2)$$

Integrating Eq. (3.2) yields,

$$(\alpha^2 - k)u' + (u')^2 + u''' = 0. \quad (3.3)$$

We suppose that constant of integration is zero. When balancing $(u')^2$ with u''' obtain $n=1$. Therefore, we may choose

$$u(\xi) = A_0 + A_1 \left[p + \tan\left(\frac{F(\xi)}{2}\right) \right] + B_1 \left[p + \tan\left(\frac{F(\xi)}{2}\right) \right]^{-1}, A_i \neq 0, B_i \neq 0. \quad (3.4)$$

Substituting Eq. (3.4) into Eq. (3.3) yields a set of algebraic equations for k, p, B_1, A_0, A_1 .

The system of equations is given as following.

$$\begin{aligned} & 2a^2 A_1 - 4A_1^2 b + 14A_1 b^2 - 2a^2 B_1 - 14b^2 B_1 + 4bB_1^2 + 6A_1^2 c - 30A_1 bc - 4A_1 B_1 c - 30bB_1 c + 6B_1^2 c \\ & + 16A_1 c - 16B_1 c - 2A_1 k + 2B_1 k + 8aA_1^2 p - 24aA_1 bp - 8aA_1 B_1 p + 24abB_1 p + 24aA_1 cp + 36aB_1 cp \\ & + 12a^2 A_1 p^2 + 12A_1 b^2 p^2 - 10a^2 B_1 p^2 - 8A_1 bB_1 p^2 + 2b^2 B_1 p^2 + 12A_1^2 cp^2 - 36A_1 bcp^2 - 12A_1 B_1 cp^2 \\ & + 6bB_1 cp^2 + 24A_1 c^2 p^2 - 8B_1 c^2 p^2 - 12A_1 kp^2 + 10B_1 kp^2 + 8aA_1^2 p^3 - 24aA_1 bp^3 + 24aA_1 cp^3 \\ & + 10a^2 A_1 p^4 + 4A_1^2 bp^4 - 2A_1 b^2 p^4 + 6A_1^2 cp^4 - 6A_1 bcp^4 + 8A_1 c^2 p^4 - 10A_1 kp^4 + 2A_1 \alpha^2 - 2B_1 \alpha^2 \\ & + 12A_1 p^2 \alpha^2 - 10B_1 p^2 \alpha^2 + 10A_1 p^4 \alpha^2 = 0, \\ & -a^2 A_1 + 7A_1^2 b - 22A_1 b^2 - a^2 B_1 - 2A_1 bB_1 - 22b^2 B_1 + \dots \end{aligned} \quad (3.5)$$

from the solutions of the system, we can find

Case1:

$$a = -bp + cp, A_1 = 0, b + c + bp^2 - cp^2 \neq 0, B_1 = 3(b + c + bp^2 - cp^2), B_1 \neq 0,$$

$$k = \frac{1}{2}(-b^2 + bB_1 - B_1c + c^2 - b^2p^2 + 2bcp^2 - c^2p^2 + 2\alpha^2), bp - cp \neq 0. \quad (3.6)$$

Case 2:

$$A_1 = 3(b - c), A_1(a + bp - cp) \neq 0, B_1 = 0, k = a^2 + b^2 - c^2 + \alpha^2, a \neq 0. \quad (3.7)$$

Case 3:

$$A_1 = 0, B_1 = 3(b + c - 2ap - bp^2 + cp^2), B_1 \neq 0, k = a^2 + b^2 - c^2 + \alpha^2, a^2 + abp - acp \neq 0. \quad (3.8)$$

Case 4:

$$a = -bp + cp, A_1 = 3(b - c), A_1 \neq 0, B_1 = 3(b + c + bp^2 - cp^2), k = b^2 + bB_1 - B_1c - c^2 + b^2p^2 - 2bcp^2 + c^2p^2 + \alpha^2, bp - cp \neq 0. \quad (3.9)$$

With the help of Wolfram Mathematica 9. Substituting Eqs. (3.6), (3.7), (3.8) and (3.9), into (3.4) we have obtained the below solution of Eq. (3.1);

Case 1

$$\text{For } a^2 + b^2 - c^2 < 0, b - c \neq 0,$$

$$u_{1.1}(x, y, t) = 3\sqrt{(c - b)(c + b + (b - c)p^2)} \text{Cot}\left[\frac{1}{2}\sqrt{(c - b)(c + b + (b - c)p^2)}(x + \alpha y - t(b^2 - c^2 + (b - c)^2p^2 + \alpha^2))\right]. \quad (3.10)$$

$$\text{For } a^2 + b^2 - c^2 > 0, b - c \neq 0,$$

$$u_{1.2}(x, y, t) = 3\sqrt{b^2 - c^2 + (bp - cp)^2} \text{Coth}\left[\frac{1}{2}\sqrt{(b - c)(c + b + (b - c)p^2)}(x + \alpha y - t(b^2 - c^2 + (b - c)^2p^2 + \alpha^2))\right]. \quad (3.11)$$

$$\text{For } a^2 + b^2 - c^2 > 0, c = 0, b \neq 0,$$

$$u_{1.3}(x, y, t) = 3b\sqrt{1 + p^2} \text{Coth}\left[\frac{1}{2}b\sqrt{1 + p^2}(x + \alpha y - t(b^2(1 + p^2) + \alpha^2))\right]. \quad (3.12)$$

For $a^2 + b^2 - c^2 < 0$, $c \neq 0$, $b = 0$,

$$u_{1.4}(x, y, t) = 3c\sqrt{1-p^2}\text{Cot}\left[\frac{1}{2}c\sqrt{1-p^2}(x + \alpha y - t(c^2(p^2 - 1) + \alpha^2))\right]. \quad (3.13)$$

For $c = a$,

$$u_{1.5}(x, y, t) = 3b - \frac{6b(p-1)}{-1 + be^{b(x+\alpha y-t(b^2+\alpha^2))} + p}. \quad (3.14)$$

For $a = c$,

$$u_{1.6}(x, y, t) = 3b + \frac{6b(p-1)}{1 + be^{b(x+\alpha y-t(b^2+\alpha^2))} - p}. \quad (3.15)$$

For $c = -a$,

$$u_{1.7}(x, y, t) = 3b\left(1 - \frac{2b}{b + e^{b(x+\alpha y-t(b^2+\alpha^2))}(1+p)}\right). \quad (3.16)$$

For $b = -c$,

$$u_{1.8}(x, y, t) = 6c\left(-1 + \frac{2c}{c + e^{2cp(4c^2p^2t-x-\alpha y+\alpha^2t)}}\right)p. \quad (3.17)$$

For $a = 0$, $b = c$,

$$u_{1.9}(x, y, t) = \frac{6c}{p + c(x + \alpha(y - \alpha t))}. \quad (3.18)$$

For $a = 0$, $b = -c$,

$$u_{1.10}(x, y, t) = -\frac{6cp^2}{p + \frac{1}{4c^3p^2t - cx - c\alpha y + c\alpha^2t}}. \quad (3.19)$$

Case 2

For $a^2 + b^2 - c^2 > 0$, $b - c \neq 0$,

$$u_{2.1}(x, y, t) = 3(a + (b - c)p + \sqrt{a^2 + b^2 - c^2} \text{Tanh}[\frac{1}{2}\sqrt{a^2 + b^2 - c^2}(x + \alpha y - t(a^2 + b^2 - c^2 + \alpha^2))]). \quad (3.20)$$

For $a^2 + b^2 - c^2 > 0, b \neq 0, c = 0$,

$$u_{2.2}(x, y, t) = 3\left(a + bp + \sqrt{a^2 + b^2} \text{Tanh}\left[\frac{1}{2}\sqrt{a^2 + b^2}(x + \alpha y - t(a^2 + b^2 + \alpha^2))\right]\right). \quad (3.21)$$

For $a^2 + b^2 - c^2 < 0, c \neq 0, b = 0$,

$$u_{2.3}(x, y, t) = 3(a - cp - \sqrt{c^2 - a^2} \text{Tan}[\frac{1}{2}\sqrt{-a^2 + c^2}(x + \alpha y - t(a^2 - c^2 + \alpha^2))]). \quad (3.22)$$

For $a^2 + b^2 = c^2$,

$$u_{2.4}(x, y, t) = 3(a + bp - \sqrt{a^2 + b^2}p + \frac{2}{x + \alpha(y - at)}). \quad (3.23)$$

For $c = a$,

$$u_{2.5}(x, y, t) = 3b + \frac{6b(p - 1)}{1 + be^{b(x + \alpha y - t(b^2 + \alpha^2))} - p}. \quad (3.24)$$

For $c = a$,

$$u_{2.6}(x, y, t) = 3(c - cp + b(1 + \frac{2}{-1 + (b - c)e^{b(x + \alpha y - t(b^2 + \alpha^2))}} + p)). \quad (3.25)$$

For $c = -a$,

$$u_{2.7}(x, y, t) = 3(a + b)(1 - \frac{2b}{a + b - e^{b(x + \alpha y - t(b^2 + \alpha^2))}} + p). \quad (3.26)$$

For $b = -c$,

$$u_{2.8}(x, y, t) = -6c(\frac{a}{-c + e^{-a(x + \alpha y) + at(a^2 + \alpha^2)}} + p). \quad (3.27)$$

For $b = 0, a = c$,

$$u_{2.9}(x, y, t) = -3c(p - 1) + \frac{6}{x + \alpha(y - \alpha t)}. \quad (3.28)$$

For $a = 0$, $b = -c$,

$$u_{2.10}(x, y, t) = -6cp + \frac{6}{x + \alpha(y - \alpha t)}. \quad (3.29)$$

For $a = 0$, $b = 0$,

$$u_{2.11}(x, y, t) = -3c(p + \text{Tan}[\frac{1}{2}c(c^2t + x + \alpha(y - \alpha t))]). \quad (3.30)$$

Case 3

For $a^2 + b^2 - c^2 > 0$, $b - c \neq 0$,

$$u_{3.1}(x, y, t) = -\frac{3(b - c)(2ap + b(p^2 - 1) - c(1 + p^2))}{a + (b - c)p + \sqrt{a^2 + b^2 - c^2}\text{Tanh}[\frac{1}{2}\sqrt{a^2 + b^2 - c^2}(x + \alpha y - t(a^2 + b^2 - c^2 + \alpha^2))]} \quad (3.31)$$

For $a^2 + b^2 - c^2 > 0$, $b \neq 0$, $c = 0$,

$$u_{3.2}(x, y, t) = -\frac{3b(2ap + b(p^2 - 1))}{a + bp + \sqrt{a^2 + b^2}\text{Tanh}[\frac{1}{2}\sqrt{a^2 + b^2}(x + \alpha y - t(a^2 + b^2 + \alpha^2))]} \quad (3.32)$$

For $a^2 + b^2 - c^2 < 0$, $c \neq 0$, $b = 0$,

$$u_{3.3}(x, y, t) = \frac{3c(c - 2ap + cp^2)}{-a + cp + \sqrt{-a^2 + c^2}\text{Tan}[\frac{1}{2}\sqrt{-a^2 + c^2}(x + \alpha y - t(a^2 - c^2 + \alpha^2))]} \quad (3.33)$$

For $a^2 + b^2 = c^2$,

$$u_{3.4}(x, y, t) = \frac{(3b - 6ap - 3bp^2 + 3\sqrt{a^2 + b^2}(1 + p^2))}{p - \frac{(\sqrt{b^2 + a^2} + b)(2 + a(x + \alpha(y - at)))}{a^2(x + \alpha(y - t\alpha))}}. \quad (3.34)$$

For $c = a$,

$$u_{3.5}(x, y, t) = \frac{3(b + a(p - 1)^2 - bp^2)}{\frac{1 - (a + b)e^{b(x + \alpha y - t(b^2 + \alpha^2))}}{-1 + (a - b)e^{b(x + \alpha y - t(b^2 + \alpha^2))}} + p}. \quad (3.35)$$

For $c = a$,

$$u_{3.6}(x, y, t) = -\frac{3(p - 1)(b + c + bp - cp)}{\frac{1 + (b + c)e^{b(x + \alpha y - t(b^2 + \alpha^2))}}{-1 + (b - c)e^{b(x + \alpha y - t(b^2 + \alpha^2))}} + p}. \quad (3.36)$$

For $c = -a$,

$$u_{3.7}(x, y, t) = \frac{3(a - b + (a + b)p)}{-1 + \frac{2b}{(a + b - e^{b(x + \alpha y - t(b^2 + \alpha^2))})(1 + p)}}. \quad (3.37)$$

For $b = -c$,

$$u_{3.8}(x, y, t) = \frac{6p(cp - a)}{\frac{a}{-c + e^{-a(x + \alpha y) + at(a^2 + \alpha^2)}} + p}. \quad (3.38)$$

For $b = 0$, $a = c$,

$$u_{3.9}(x, y, t) = \frac{3c^2(p - 1)^2(x + \alpha(y - at))}{-2 + c(p - 1)(x + \alpha(y - at))}. \quad (3.39)$$

For $a = 0$, $b = c$,

$$u_{3.10}(x, y, t) = \frac{6c}{p + c(x + \alpha(y - at))}. \quad (3.40)$$

For $a = 0$, $b = -c$,

$$u_{3.11}(x, y, t) = \frac{6cp^2}{p - \frac{1}{c(x + \alpha(y - \alpha t))}}. \quad (3.41)$$

For $a = 0, b = 0$,

$$u_{3.12}(x, y, t) = \frac{3c(1 + p^2)}{p + \tan[\frac{1}{2}c(c^2t + x + \alpha(y - \alpha t))]} . \quad (3.42)$$

Case 4

For $a^2 + b^2 - c^2 > 0, b - c \neq 0$,

$$u_{4.1}(x, y, t) = 6\sqrt{b^2 - c^2 + (bp - cp)^2} \coth \left[\sqrt{b^2 - c^2 + (bp - cp)^2} (x + \alpha y - t(4(b - c)(b + c + (b - c)p^2) + \alpha^2)) \right]. \quad (3.43)$$

For $a^2 + b^2 - c^2 < 0, c \neq 0, b = 0$,

$$u_{4.2}(x, y, t) = 6c\sqrt{1 - p^2} \cot[c\sqrt{1 - p^2}(-4c^2(p^2 - 1)t + x + \alpha(y - \alpha t))] . \quad (3.44)$$

For $c = a$,

$$u_{4.3}(x, y, t) = \frac{6b}{1 - \frac{2(p - 1)^2}{b^2 e^{2b(-4b^2t + x + \alpha(y - \alpha t))}} + (p - 1)^2}. \quad (3.45)$$

For $c = a$,

$$u_{4.4}(x, y, t) = \frac{6b}{1 - \frac{2(p - 1)^2}{b^2 e^{2b(-4b^2t + x + \alpha(y - \alpha t))}} + (p - 1)^2}. \quad (3.46)$$

For $b = 0, a = c$,

$$u_{4.5}(x, y, t) = \frac{6}{x + \alpha(y - \alpha t)}. \quad (3.47)$$

For $a = 0, b = c$,

$$u_{4.6}(x, y, t) = \frac{6c}{p + c(x + \alpha(y - \alpha t))}. \quad (3.48)$$

For $a = 0, b = 0$,

$$u_{4.7}(x, y, t) = -3cp - 3c \tan\left[\frac{1}{2}c(-4c^2(p^2 - 1)t + x + \alpha(y - \alpha t))\right] - \frac{3c(p^2 - 1)}{p + \tan\left[\frac{1}{2}c(-4c^2(p^2 - 1)t + x + \alpha(y - \alpha t))\right]}. \quad (3.49)$$

Example2: Consider the (2+1)-dimensional Calogero–Bogoyavlenskii–Schiff equations [27].

$$u_{xxxy} + 2u_y u_{xx} + 4u_x u_{xy} + u_{xt} = 0. \quad (3.50)$$

Let us concentrate the traveling wave solutions, $u(\xi) = u(x, y, t)$, $u(\xi) = x + \alpha y - kt$ then Eq. (3.1) becomes

$$\alpha u^{(4)} + 6\alpha u' u'' - k u'' = 0. \quad (3.51)$$

And integrating (3.2) yields,

$$\alpha u''' + 3\alpha (u')^2 - kt = 0. \quad (3.52)$$

When balancing $(u')^2$ with u''' then gives $n=1$ Therefore, we may choose

$$u(\xi) = A_0 + A_1 \left[p + \tan\left(\frac{F(\xi)}{2}\right) \right] + B_1 \left[p + \tan\left(\frac{F(\xi)}{2}\right) \right]^{-1}, A_i \neq 0, B_i \neq 0. \quad (3.53)$$

Substituting (3.53) into Eq. (3.52) yields a set of algebraic equations for k, p, A_0, A_1, B_1 . these systems are finding as

$$\begin{aligned} & -2A_1 k + 2B_1 k - 12A_1 k p^2 + 10B_1 k p^2 - 10A_1 k p^4 + 2a^2 A_1 \alpha - 12A_1^2 b \alpha + 14A_1 b^2 \alpha - \\ & 2a^2 B_1 \alpha - 14b^2 B_1 \alpha + 12b B_1^2 \alpha + 18A_1^2 c \alpha - 30A_1 b c \alpha - 12A_1 B_1 c \alpha - 30b B_1 c \alpha + \\ & 18B_1^2 c \alpha + 16A_1 c^2 \alpha - 16B_1 c^2 \alpha + 24a A_1^2 p \alpha - 24a A_1 b p \alpha - 24a A_1 B_1 p \alpha + \\ & 24ab B_1 p \alpha + 24a A_1 c p \alpha + 36a B_1 c p \alpha + 12a^2 A_1 p^2 \alpha + 12A_1 b^2 p^2 \alpha - 10a^2 A_1 p^2 \alpha - \\ & 24A_1 b B_1 p^2 \alpha + 2b^2 A_1 p^2 \alpha + 36A_1^2 c p^2 \alpha - 36A_1 b c p^2 \alpha - 36A_1 B_1 c p^2 \alpha + 6b B_1 c p^2 \alpha + \\ & 24A_1 c^2 p^2 \alpha - 8B_1 c^2 p^2 \alpha + 24a A_1^2 p^3 \alpha - 24a A_1 b p^3 \alpha + 24a A_1 c p^3 \alpha + 10a^2 A_1 p^4 \alpha + \\ & 12A_1^2 b p^4 \alpha - 2A_1 b^2 p^4 \alpha + 18A_1^2 c p^4 \alpha - 6A_1 b c p^4 \alpha + 8A_1 c^2 p^4 \alpha = 0 \end{aligned}$$

$$8B_1kp - 16A_1kp^3 - 12aA_1^2\alpha + 12aA_1b\alpha \dots \dots \dots (3.54)$$

From the solutions of the system, we can found:

Case1:

$$(A_1 = 0, B_1 = b + c - 2ap - bp^2 + cp^2, B_1 \neq 0, k = (a^2 + b^2 - c^2)\alpha, a^2\alpha + abp\alpha - acp\alpha \neq 0). (3.55)$$

Case 2:

$$(A_1 = b - c, A_1(a + bp - cp) \neq 0, B_1 = 0, k = (a^2 + b^2 - c^2)\alpha, a\alpha \neq 0). (3.56)$$

Case 3:

$$(a = -bp + cp, A_1 = b - c, A_1 \neq 0, (B_1 = 0 \mid |B_1 = b + c + bp^2 - cp^2), k = (b - c)(b + 3B_1 + c + bp^2 - cp^2)\alpha, bpa - cp\alpha \neq 0). (3.57)$$

Case 4:

$$. (a = -bp + cp, A_1 = 0, b + c + bp^2 - cp^2 \neq 0, B_1 = b + c + bp^2 - cp^2, B_1 \neq 0, k = -\frac{1}{2}(b - c)(b - 3B_1 + c + bp^2 - cp^2)\alpha, bpa - cp\alpha \neq 0). (3.58)$$

Case 1

For $a^2 + b^2 - c^2 < 0, b - c \neq 0$,

$$\begin{aligned} &u_{1.1}(x, y, t) \\ &= -\frac{(b - c)(2ap + b(-1 + p^2) - c(1 + p^2))}{a + (b - c)p - \sqrt{c^2 - b^2 - a^2}\text{Tan}[\frac{1}{2}\sqrt{c^2 - b^2 - a^2}(x + ((c^2 - b^2 - a^2)t + y)\alpha)]}. \end{aligned} (3.59)$$

For $a^2 + b^2 - c^2 > 0, b - c \neq 0$,

$$\begin{aligned} &u_{1.2}(x, y, t) \\ &= -\frac{(b - c)(2ap + b(-1 + p^2) - c(1 + p^2))}{a + (b - c)p + \sqrt{a^2 + b^2 - c^2}\text{Tanh}[\frac{1}{2}\sqrt{b^2 + a^2 - c^2}(x + ((c^2 - b^2 - a^2)t + y)\alpha)]}. \end{aligned} (3.60)$$

For $a^2 + b^2 - c^2 > 0, b \neq 0, c = 0$,

$$u_{1.3}(x, y, t) = - \frac{(b - c)(2ap + b(-1 + p^2) - c(1 + p^2))}{a + (b - c)p + \sqrt{a^2 + b^2 - c^2} \operatorname{Tanh}[\frac{1}{2}\sqrt{a^2 + b^2 - c^2}(x + (-(a^2 + b^2 - c^2)t + y)\alpha)]}. \quad (3.61)$$

For $a^2 + b^2 - c^2 < 0$, $c \neq 0$, $b = 0$,

$$u_{1.4}(x, y, t) = \frac{c(c - 2ap + cp^2)}{-a + cp + \sqrt{-a^2 + c^2} \operatorname{Tan}[\frac{1}{2}\sqrt{-a^2 + c^2}(x + (-a^2t + c^2t + y)\alpha)]}. \quad (3.62)$$

For $a^2 + b^2 = c^2$,

$$u_{1.5}(x, y, t) = \frac{b - 2ap - bp^2 + \sqrt{a^2 + b^2}(1 + p^2)}{p - \frac{(b + \sqrt{a^2 + b^2})(2 + a(x + \alpha y))}{a^2(x + \alpha y)}}. \quad (3.63)$$

For $c = a$,

$$u_{1.6}(x, y, t) = \frac{b + a(-1 + p)^2 - bp^2}{\frac{1 - (a + b)e^{b(x + (-b^2t + y)\alpha)}}{-1 + (a - b)e^{b(x + (-b^2t + y)\alpha)}} + p}. \quad (3.64)$$

For $a = c$,

$$u_{1.7}(x, y, t) = - \frac{(-1 + p)(b + c + bp - cp)}{\frac{1 + (c + b)e^{b(x + (-b^2t + y)\alpha)}}{-1 + (b - c)e^{b(x + (-b^2t + y)\alpha)}} + p}. \quad (3.65)$$

For $c = -a$,

$$u_{1.8}(x, y, t) = \frac{a - b + (b + a)p}{-1 + \frac{2b}{(a + b - e^{b(x - ab^2t + \alpha y))}(1 + p)}}. \quad (3.66)$$

For $b = -c$,

$$u_{1.9}(x, y, t) = \frac{2p(-a + cp)}{\frac{a}{-c + e^{aa^3t - a(x + \alpha y)}} + p}. \quad (3.67)$$

For $b = 0, a = c$,

$$u_{1.10}(x, y, t) = \frac{c^2(-1+p)^2(x+\alpha y)}{-2+c(-1+p)(x+\alpha y)}. \quad (3.68)$$

For $a = 0, b = c$,

$$u_{1.11}(x, y, t) = \frac{2c}{p+c(x+\alpha y)}. \quad (3.69)$$

For $a = 0, b = -c$,

$$u_{1.12}(x, y, t) = 2cp \left(1 + \frac{1}{-1+cp(x+\alpha y)} \right). \quad (3.70)$$

For $a = 0, b = 0$,

$$u_{1.13}(x, y, t) = \frac{c(1+p^2)}{p + \text{Tan}[\frac{1}{2}c(x+(c^2t+y)\alpha)]}. \quad (3.71)$$

Case 2

For $a^2 + b^2 - c^2 < 0, b - c \neq 0$,

$$u_{2.1}(x, y, t) = a + (b - c)p - \sqrt{-a^2 - b^2 + c^2} \text{Tan}[\frac{1}{2}\sqrt{-b^2 - a^2 + c^2}(x + ((c^2 - b^2 - a^2)t + y)\alpha)]. \quad (3.72)$$

For $a^2 + b^2 - c^2 > 0, b - c$,

$$u_{2.2}(x, y, t) = a + (b - c)p + \sqrt{a^2 + b^2 - c^2} \text{Tanh}[\frac{1}{2}\sqrt{b^2 + a^2 - c^2}(x + ((c^2 - a^2 - b^2)t + y)\alpha)]. \quad (3.73)$$

For $a^2 + b^2 - c^2 > 0, b \neq 0, c = 0$,

$$u_{2.3}(x, y, t) = a + bp + \sqrt{a^2 + b^2} \text{Tanh} \left[\frac{1}{2}\sqrt{a^2 + b^2}(x + (-(a^2 + b^2)t + y)\alpha) \right]. \quad (3.74)$$

For $a^2 + b^2 - c^2 > 0, c \neq 0, b = 0$,

$$u_{2.4}(x, y, t) = a - cp - \sqrt{-a^2 + c^2} \tan \left[\frac{1}{2} \sqrt{-a^2 + c^2} (x + (-a^2 t + c^2 t + y) \alpha) \right]. \quad (3.75)$$

$$\text{For } a^2 + b^2 = c^2,$$

$$u_{2.5}(x, y, t) = a + bp - \sqrt{a^2 + b^2} p + \frac{2}{x + \alpha y}. \quad (3.76)$$

$$\text{For } c = a,$$

$$u_{2.6}(x, y, t) = a - ap + b \left(1 - \frac{2}{1 + (-a + b) e^{b(x - ab^2 t + \alpha y)}} + p \right). \quad (3.77)$$

$$\text{For } a = c,$$

$$u_{2.7}(x, y, t) = c - cp + b \left(1 + \frac{2}{-1 + (b - c) e^{b(x - b^2 t \alpha + \alpha y)}} + p \right). \quad (3.78)$$

$$\text{For } c = -a,$$

$$u_{2.8}(x, y, t) = (a + b) \left(1 - \frac{2b}{a + b - e^{b(x - ab^2 t + \alpha y)}} + p \right). \quad (3.79)$$

$$\text{For } b = -a,$$

$$u_{2.9}(x, y, t) = -2c \left(\frac{a}{-c + \alpha e^{a^3 t - a(x + \alpha y)}} + p \right). \quad (3.80)$$

$$\text{For } b = 0, a = c,$$

$$u_{2.10}(x, y, t) = c - cp + \frac{2}{x + \alpha y}. \quad (3.81)$$

$$\text{For } a = 0, b = -c,$$

$$u_{2.11}(x, y, t) = -2cp + \frac{2}{x + \alpha y}. \quad (3.82)$$

$$\text{For } a = 0, b = 0,$$

$$u_{2.12}(x, y, t) = -c \left(p + \tan \left[\frac{1}{2} c (x + (c^2 t + y) \alpha) \right] \right). \quad (3.83)$$

Case 3

For $a^2 + b^2 - c^2 < 0$, $b - c \neq 0$,

$$u_{3.1}(x, y, t) = \sqrt{(c-b)(c+b+(b-c)p^2)} \left[-1 + \text{Cot} \left[\frac{1}{2} \sqrt{-(b-c)(b+c+(b-c)p^2)} (x + ((-4b+4c)(c+b+(b-c)p^2)t+y)\alpha) \right]^2 \right]. \quad (3.84)$$

For $a^2 + b^2 - c^2 > 0$, $b - c \neq 0$,

$$u_{3.2}(x, y, t) = \sqrt{b^2 - c^2 + (bp - cp)^2} \left[1 + \text{Coth} \left[\frac{1}{2} \sqrt{(b-c)(b+c+(b-c)p^2)} (x + (-4(b-c)(b+c+(b-c)p^2)t-y)\alpha) \right]^2 \right]. \quad (3.85)$$

For $a^2 + b^2 - c^2 > 0$, $b \neq 0$, $c = 0$,

$$u_{3.3}(x, y, t) = 2b\sqrt{1+p^2} \text{Coth} \left[b\sqrt{1+p^2} (x + (-4b^2(1+p^2)t+y)\alpha) \right]. \quad (3.86)$$

For $a^2 + b^2 - c^2 > 0$, $c \neq 0$, $b = 0$,

$$u_{3.4}(x, y, t) = 2c\sqrt{1-p^2} \text{Cot} \left[c\sqrt{1-p^2} (x + (-4c^2(-1+p^2)t+y)\alpha) \right]. \quad (3.87)$$

For $c = a$,

$$u_{3.5}(x, y, t) = \frac{2b}{1 - \frac{2(-1+p)^2}{b^2 e^{2b(x+(-4b^2t+y)\alpha)} + (-1+p)^2}}. \quad (3.88)$$

For $a = c$,

$$u_{3.6}(x, y, t) = \frac{2b}{1 - \frac{2(-1+p)^2}{b^2 e^{2b(x+(-4b^2t+y)\alpha)} + (-1+p)^2}}. \quad (3.89)$$

For $a = -c$,

$$u_{3.7}(x, y, t) = -\frac{2b(b^2 + e^{2b(x+(-4b^2t+y)\alpha)}(1+p)^2)}{b^2 - e^{2b(x+(-4b^2t+y)\alpha)}(1+p)^2}. \quad (3.90)$$

Case 4

For $a^2 + b^2 - c^2 < 0$, $b - c \neq 0$,

$$u_{4.1}(x, y, t) = \sqrt{(c - b)(c + b + (bp^2 - cp^2))} \text{Cot}\left[\frac{1}{2}\sqrt{(c - b)(+b + c(b - c)p^2)}(x + (-b - c)(c + b + (b - c)p^2)t + y)\alpha\right]. \quad (3.91)$$

For $a^2 + b^2 - c^2 > 0$, $b - c \neq 0$,

$$u_{4.2}(x, y, t) = \sqrt{b^2 - c^2 + (bp - cp)^2} \text{Coth}\left[\frac{1}{2}\sqrt{(b - c)(b + c + (b - c)p^2)}(x + ((c - b)(b + c + (b - c)p^2)t + y)\alpha)\right]. \quad (3.92)$$

For $a^2 + b^2 - c^2 > 0$, $b \neq 0$, $c = 0$,

$$u_{4.3}(x, y, t) = b\sqrt{1 + p^2} \text{Coth}\left[\frac{1}{2}b\sqrt{1 + p^2}(x + (-b^2(1 + p^2)t + y)\alpha)\right]. \quad (3.93)$$

For $a^2 + b^2 - c^2 > 0$, $c \neq 0$, $b = 0$,

$$u_{4.4}(x, y, t) = c\sqrt{1 - p^2} \text{Cot}\left[\frac{1}{2}c\sqrt{1 - p^2}(x + c(c - cp^2)at + \alpha y)\right]. \quad (3.94)$$

For $c = a$,

$$u_{4.5}(x, y, t) = b - \frac{2b(-1 + p)}{-1 + be^{b(x + (-b^2t + y)\alpha)} + p}. \quad (3.95)$$

For $a = c$,

$$u_{4.6}(x, y, t) = b + \frac{2b(-1 + p)}{1 + be^{b(x + (-b^2t + y)\alpha)} - p}. \quad (3.96)$$

For $b = -c$,

$$u_{4.7}(x, y, t) = 2c \left(1 - \frac{2}{1 + ce^{2cp(x + (-4c^2p^2t + y)\alpha)}}\right)p. \quad (3.97)$$

For $b = 0$, $a = c$,

$$u_{4.8}(x, y, t) = \frac{2c}{1 + c(x + \alpha y)}. \quad (3.98)$$

4. GRAPHICS

In this section we give some 2-dimensional and 3-dimensional plots of some solutions.

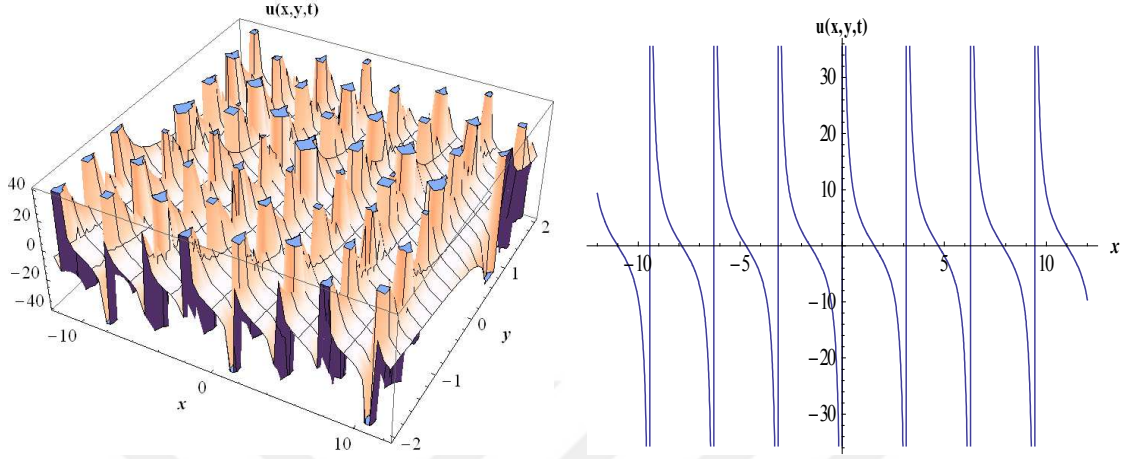


Figure 1: The 3-D and 2-D surfaces of Eq. (3.10) by using the values $\alpha = 0.7$, $p = 1$, $b = 2$, $c = 3$, $t = 0.002$, $-12 < x < 12$, $-2 < y < 2$ and $y = 0.001$ for 2D .

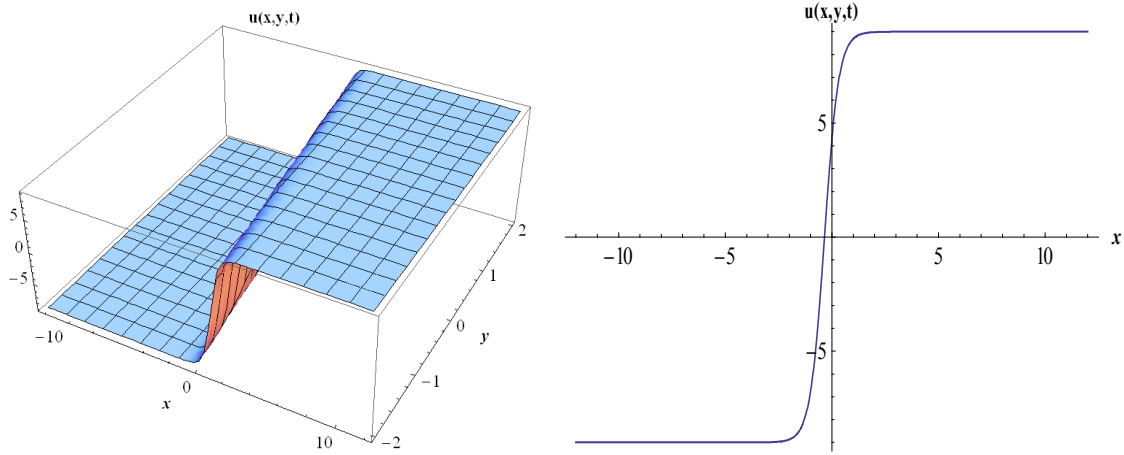


Figure 2: The 3-D and 2-D surfaces of Eq. (3.17) by using the values $\alpha = 0.7$, $p = 0.5$, $c = 3$, $t = 0.002$, $-12 < x < 12$, $-2 < y < 2$ and $y = 0.001$ for 2D.

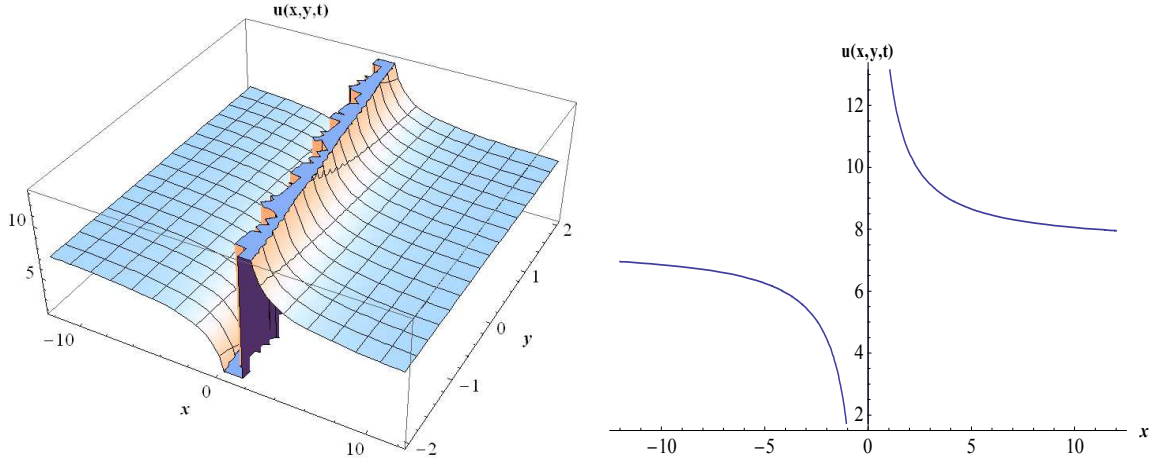


Figure 3: The 3-D and 2-D surfaces of Eq. (3.23) by using the values $\alpha = 0.7$, $p = 0.5$, $a = 3.5$, $b = 2$, $t = 0.002$, $-12 < x < 12$, $-2 < y < 2$ and $y = 0.001$ for 2D .

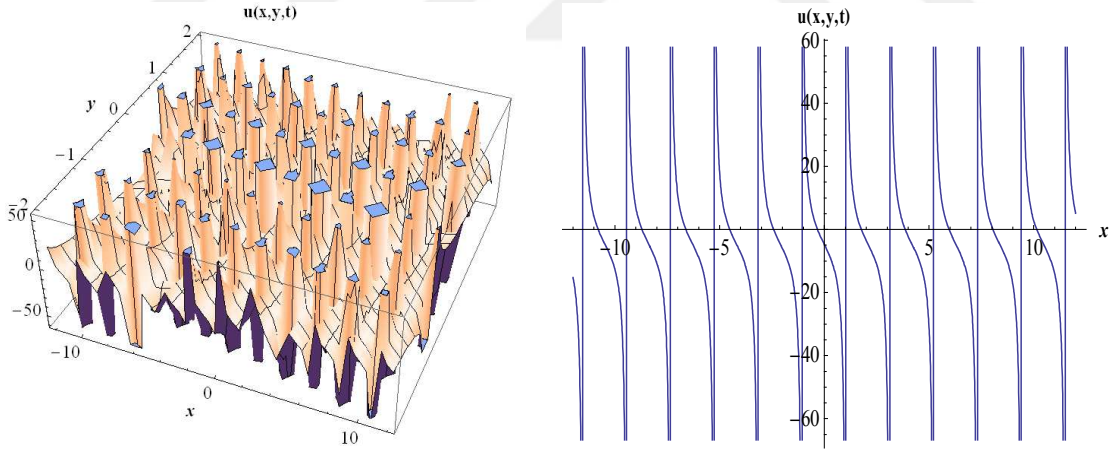


Figure 4: The 3-D and 2-D surfaces of Eq. (3.30) by using the values $\alpha = 0.7$, $p = 0.5$, $c = 3$, $t = 0.002$, $-12 < x < 12$, $-2 < y < 2$ and $y = 0.001$ for 2D .

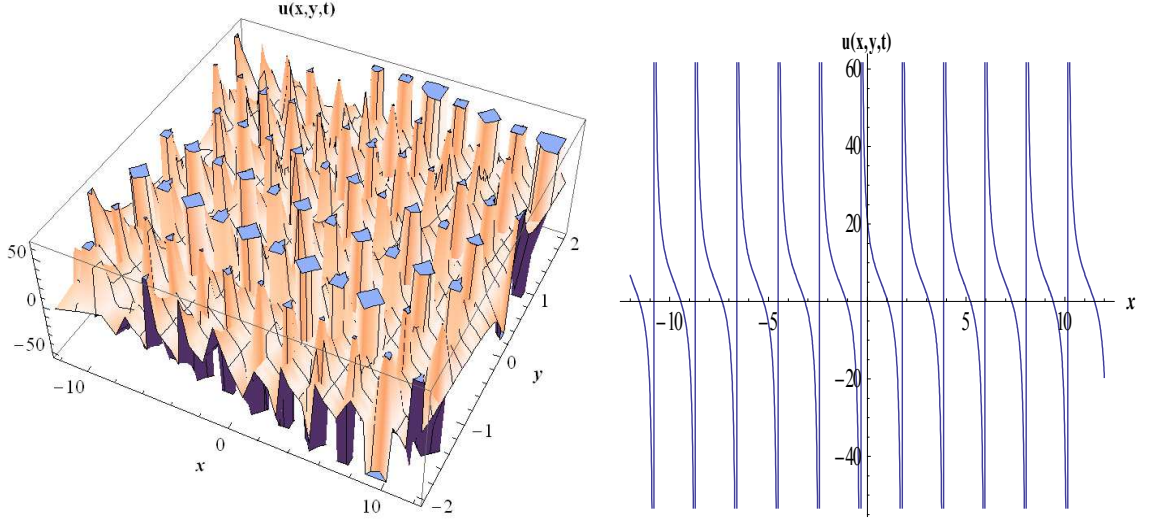


Figure 5: The 3-D and 2-D surfaces of Eq. (3.42) by using the values
 $\alpha = 0.7$, $p = 0.5$, $c = 3$, $t = 0.002$, $-12 < x < 12$, $-2 < y < 2$ and
 $y = 0.001$ for 2D .

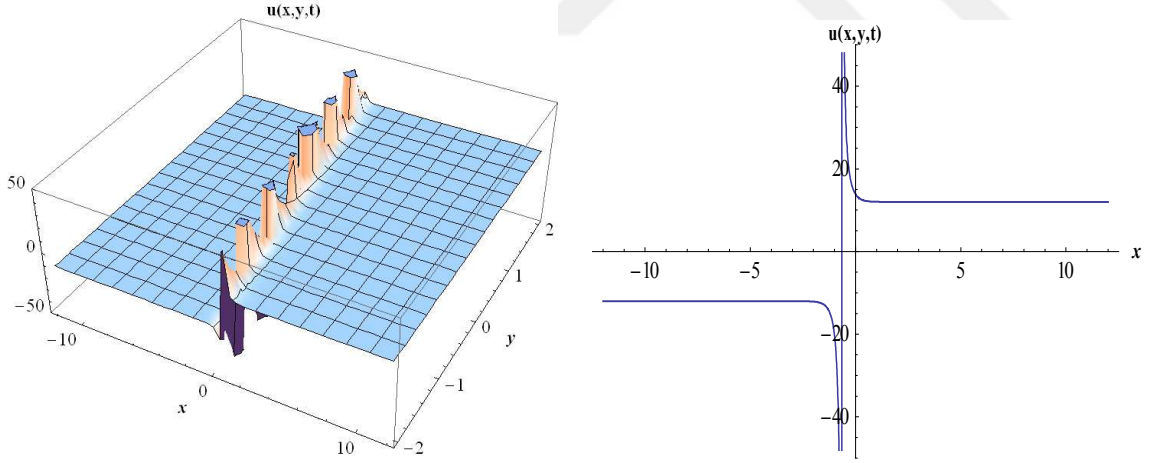


Figure 6: The 3-D and 2-D surfaces of Eq. (3.45) by using the values
 $\alpha = 0.7$, $p = 0.5$, $b = 2$, $t = 0.002$, $-12 < x < 12$,
 $-2 < y < 2$ and $y = 0.001$ for 2D .

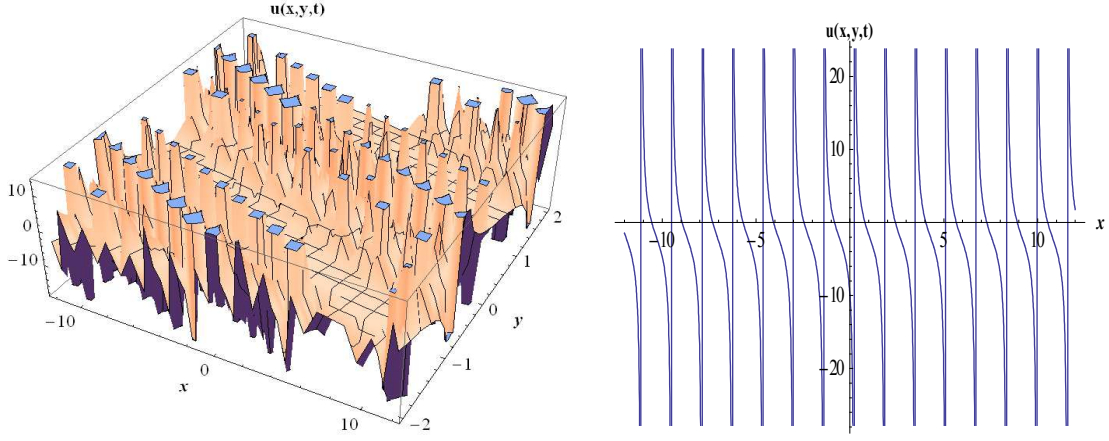


Figure 7: The 3-D and 2-D surfaces of Eq. (3.60) by using the values $\alpha = 0.7$, $p = 0.5$, $b = 5$, $c = 7$, $t = 0.002$, $-12 < x < 12$, $-2 < y < 2$ and $y = 0.003$ for 2D .

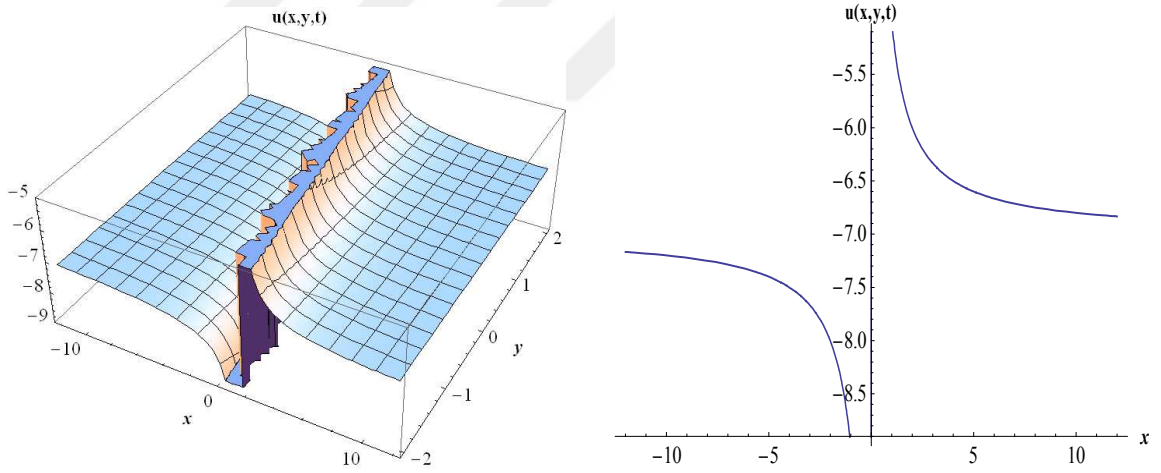


Figure 8: The 3-D and 2-D surfaces of Eq. (3.82) by using the values $\alpha = 0.7$, $p = 0.5$, $b = 5$, $c = 7$, $t = 0.002$, $-12 < x < 12$, $-2 < y < 2$ and $y = 0.003$ for 2D .

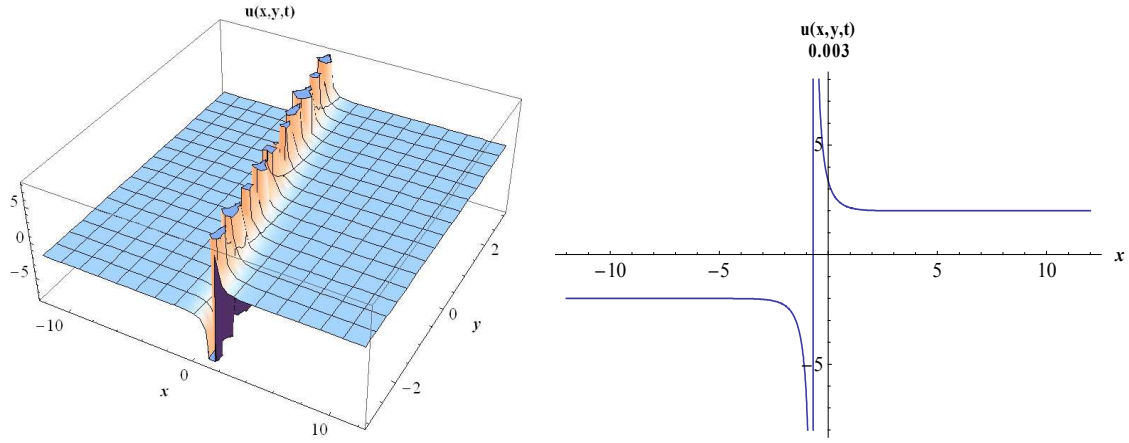


Figure 9: The 3-D and 2-D surfaces of Eq. (3.95) by using the values

$\alpha = 0.7, p = 0.5, b = 2, t = 0.002, -12 < x < 12, -3 < y < 3$ and $y = 0.003$ for 2D .

5. CONCLUSIONS

In this study, we apply $\tan\left(\frac{F(\xi)}{2}\right)$ -expansion method to the potential KdV and Calogero–Bogoyavlenskii–Schiff equations. We construct new solutions to the equation in form of trigonometric function and rational function by using Wolfram Mathematica 9 program. We also plot the two- and three dimensional graphics to some obtained solutions using the same program in Wolfram Mathematica 9. All the obtained solution are verified to be the solution to eq.(3.1) and eq.(3.59) also using the same program. With this we can say that $\tan\left(\frac{F(\xi)}{2}\right)$ -expansion method is easy and computerizable method that can be applied to various NPDEs.

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