DEFERRED STATISTICAL CONVERGENCE OF ORDER α

Omer Mohammed OMER

Master Thesis

Department: Mathematics

Program: Analysis and Functions Theory

Supervisor: Prof. Dr. Mikail ET

ELAZIG – 2017

REPUBLIC OF TURKEY

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LIST OF CONTENT

SUMMARY

In the first part of this thesis is consisting of five chapters, some basic concepts related to the subject are given.

In the second part, deferred statistical convergence and strong deferred Cesaro convergence are investigated and the relations between deferred statistical convergence and strong deferred Cesaro convergence are given.

In the third part, statistical convergence of order α and strong Cesaro convergence of order α are examined and the relations between these concepts are given.

In the fourth part statistical boundedness of order α was defined and the relations between statistical boundedness of order α. and statistical boundedness are given.

In the last part deferred statistical convergence of order α. and strong deferred Cesaro convergence of order α. were defined and the relations between these concepts are given.

ÖZET

α. Dereceden Deferred Istatistiksel Yakınsaklık

Beş bölümden oluşan bu tezin ilk bölümünde konuya ilişkin bazı temel kavramlar verilmiştir.

İkinci bölümde deferred istatistiksel yakınsaklık ve kuvvetli deferred Cesaro yakınsaklık incelenmiş ve bu kavramlar arasındaki ilişki verilmiştir.

Üçüncü bölümde α. dereceden istatistiksel yakınsaklık ve α. dereceden Cesaro yakınsaklık incelenmiş ve bu kavramlar arasındaki ilişki verilmiştir.

Dördüncü bölümde α. dereceden istatistiksel sınırlılık tanımlanmış ve α. dereceden istatistiksel sınırlılık ile istatistiksel sınırlılık arasındaki ilişki verilmiştir.

Son bölümde α. dereceden deferred istatistiksel yakınsaklık ve α. dereceden kuvvetli deferred Cesaro yakınsaklık tanımlanmış ve bu kavramlar arasındaki ilişki verilmiştir.

LIST OF SYMBOLS

$\mathbb{N}$	Set of the natural numbers.
$\mathbb{R}$	Set of the real numbers.
$\mathbb{C}$	Set of the complex numbers.
$a.a.k$	almost all k
ℓ_∞	space of bounded sequences
$\in$	belongs to
$\notin$	dose not belong to
$i.e.$	it means
S	Set of statistically convergence
S^α	Set of statistically convergence of order α
S_d^α	Set of deferred statistically convergence of order α
$S(b)$	Set of set of statistically bounded
$S^\alpha(b)$	Set of statistically bounded of order α
$S_d^\alpha(b)$	Set of deferred statistically bounded of order α

1. INTRODUCTION, DEFINITIONS AND PRELIMINARIES

The idea of statistical convergence was given by Zygmund [1] in the first edition of his monograph published in Warsaw in 1935 and was firstly considered as a summability method by Schoenberg [2] in 1959 despite the fact that it made its initial appearance in a short note by Fast [3] and Steinhaus [4] in 1951. Along with the theory of summability it has played an important role in Fourier analysis, Ergodic theory, Number theory, Measure theory, Trigonometric series, Turnpike theory and Banach spaces thanks to many distinguished authors' contributions. For instance, the concept appeared to have been an example of convergence in density introduced by Buck [5] in 1953. Salat [6] also showed that the set of bounded statistical convergent real valued sequences is a closed subspace of bounded sequences in 1980. After that Fridy [7] introduced the concept of statistically Cauchiness of sequences and proved that it is equivalent to statistical convergence. More importantly he proved that there is no any matrix summability method which involves statistical convergence in the same paper in 1985. Another magnificent development was presented to the literature by Connor [8] in 1988. For the first time in the literature his work confirmed the direct link between statistical convergence and strong p-Cesaro summability by unveiling that the notions are equivalent for bounded sequences. Beside he showed that the set of statistically convergent sequences does not generate a locally convex FK space.

Recently, generalizations of statistical convergence have started to arise in many articles by several authors. Mursaleen [9] gave the concept of λ-statistical convergence in 2000, while Savaş [10] examined the relationship between strong almost convergence and almost λ-statistical convergence in the same year. Et and Nuray [11] firstly associated the difference sequences of order m and statistical convergence by introducing Δ^m-statistical convergence. Afterwards Çolak [12] made a new approach to the concept by studying the idea of statistical convergence of order α ,where $0 < \alpha \leq 1$, in 2010. Nowadays statistical convergence have been studied by many mathematicians such as Bhardwaj et al. [13-14], Çolak [15], Çolak and Bektaş [16], Connor [17], Fridy [18], Fridy and Orhan ([19],[20]), Et et al. ([21-24]), Gadjiev and Orhan [25], Isik ([26-27]), Mursaleen *et al.* ([28-30]), Savaş [31-33], Rath and Tripathy [34], Temizsu et al. [35] and many others.

In 1932 Agnew [36] presented the deffered Cesaro mean by modifying Cesaro mean to obtain more useful metods including stronger features which do not belong to nearly all methods. Küçükaslan and Yılmaztürk [37-38] came up with the idea of combining the deferred Cesaro mean and the concept of statistical convergence. This gave them the opportunity to generalise both strong p-Cesaro summability and statistical convergence with the sense of deferred Cesaro mean.

Let $K \subset \mathbb{N}$, the natural density of Kdefined by

$$\delta(K) = \lim_{i\to\infty} \frac{1}{i} |\{j \leq i : j \in K\}| \ .$$

A sequence x is called to be statistically convergent to σ if for each $\varepsilon > 0$,

$$\delta(\{i \in \mathbb{N}: |x_i - \sigma| > \varepsilon\}) = 0 \ .$$

In this situation, we write $st - lim_i x_i = \sigma$. The set of all statistically convergent sequences will be denoted by S.

In 1932, Agnew [36] defined deferred Cesàro mean of sequences of real numbers such as:

$$(Dx)_i = \frac{1}{s_i - t_i} \sum_{t_i+1}^{s_i} x_j \qquad \textbf{(1.1)}$$

where (t_i) and (s_i) are two sequences of non-negative integers satisfying

$$t_i < s_i \text{ and } \lim_{i\to\infty} s_i = +\infty. \qquad \textbf{(1.2)}$$

Let $K \subset \mathbb{N}$ and, set $\mathrm{K}(i) = \{j: t_i < j \leq s_i,\ j \in K\}$. We can define the deferred α-density of K defined by

$$\delta^\alpha(K) = \lim_{i\to\infty} \frac{1}{(s_i - t_i)^\alpha} |\mathrm{K}(i)|, \text{ for } 0 < \alpha \leq 1 \ , \qquad \textbf{(1.3)}$$

where the vertical bars show the cardinality of the set K. We can define upper α-deferred asymptotic density of K defined by

$$\bar{\delta}^{\alpha}(K) = \lim_{i\to\infty} sup \frac{1}{(s_i - t_i)^{\alpha}} |\mathrm{K}(i)|.$$

It is clear that

i) If $\delta^{\alpha}(K)$ is exists then $\delta^{\alpha}(K) = \bar{\delta}^{\alpha}(K)$,

ii) If $\delta^{\alpha}(K) \neq 0$ then $\bar{\delta}^{\alpha}(K) \geq 0$,

iii) If $K \subset M$, then $\bar{\delta}^{\alpha}(K) \leq \bar{\delta}^{\alpha}(M)$,

iv) The deferred α −density with coincides α −density of K, if $s_i = i, t_i = 0$,

v) Let $\alpha, \beta \in (0,1]$, then $\delta^{\alpha}(K) \leq \delta^{\beta}(K)$,

The sequence x is called to be λ-statistically convergent or S_{λ}-convergent to σ if for each $\varepsilon > 0$

$$\lim_{i\to\infty} \frac{1}{\lambda_i} \left|\left\{ j \in I_i : \left|x_j - \sigma\right| > \varepsilon\right\}\right| = 0,$$

while $I_i = [i - \lambda_i + 1, i]$ and $\lambda_{i+1} \leq \lambda_i + 1, \lambda_1 = 1$. By Λ and S_{λ} , we will denote the set of all such sequences (λ_i) and the set of all λ-statistically convergent sequences, respectively.

In 2010, Çolak [12] defined a new type statistical convergence by specifying the α-density $\delta_{\alpha}(K)$, ($0 < \alpha \leq 1$) of a subset K of $\mathbb{N}$ as:

$$\delta_{\alpha}(K) = \lim_{i\to\infty} \frac{1}{i^{\alpha}} |\{j \leq i : j \in K\}|,$$

Let $\alpha \in (0,1]$. A sequence x is said to be statistically convergent of order α if the following equality is satisfied

$\lim_{i\to\infty} \frac{1}{i^{\alpha}} \left|\left\{j \leq i : \left|x_j - \sigma\right| \geq \varepsilon\right\}\right| = 0.$

For the set of all statistically convergent sequences of order α, we shall write S^{α} and we shall write S_0^{α} instead of S^{α}, when σ =0.

It's clear $c \subset S^{\alpha}$ for every $0 < \alpha \leq 1$. For $\alpha > 1$, statistical convergence of order α isn't well defined. Really let x be the following sequence:

$$x_j = \begin{cases} 1, & j = 2i \\ 0, & j \neq 2i \end{cases} \quad i = 1,2,3, \ldots$$

Then, both

$$\lim_{i\to\infty} \frac{1}{i^{\alpha}} \left|\left\{j \leq i: \left|x_j - 1\right| \geq \varepsilon\right\}\right| \leq \lim_{i\to\infty} \frac{i}{2i^{\alpha}} = 0$$

and

$$\lim_{i\to\infty} \frac{1}{i^{\alpha}} \left|\left\{j \leq i: \left|x_j\right| \geq \varepsilon\right\}\right| \leq \lim_{i\to\infty} \frac{i}{2i^{\alpha}} = 0$$

for $\alpha > 1$. So $S^{\alpha}\text{–} \lim x_i = 1$ and $S^{\alpha}\text{–} \lim x_i = 0$. But this is not possible.

In 2011, Çolak and Bektaş [16] defined the concept of λ-statistical convergence of order α, such as:

A sequence x is called to be λ-statistically convergent of order α to σ, if the following equality is satisfied:

$$\lim_{i\to\infty} \frac{1}{\lambda_i^{\alpha}} \left|\left\{j \in I_i: \left|x_j - \sigma\right| > \varepsilon\right\}\right| = 0,$$

where $I_i = [i - \lambda_i + 1, i]$ and $0 < \alpha \leq 1$. We will write S_{λ}^{α}, for the set of all sequences which are λ statistically convergent of order α.

Let x be any sequence space, them

1) X is normal, if $y = (y_i) \in X$i, when $|y_i| \leq |x_i|$, $i \geq 1$, for some $x \in X$,

2) X is monotone as long as it has the canonical preimages of all its stepspaces,

3) X is sequence algebra if $xy = (x_i y_i) \in X$ whenever $x, y \in X$,

4) Symmetric if $(x_i) \in X$ implies $(x_{\pi(i)}) \in X$ when π is a permutation on $\mathbb{N}$.

Let X be any sequence space and define

$$X^{\alpha} = \left\{ a = (a_i) \colon \sum_i |a_i x_i| < \infty \text{ for all } x = (x_i) \in X \right\},$$

$$X^{\beta} = \left\{ a = (a_i) \colon \sum_k a_i x_i \text{ is convergent for all } x = (x_i) \in X \right\},$$

then X^{α} and X^{β} are called α- and β-duals of X respectively. Obviously $\phi \subset X^{\alpha} \subset X^{\beta}$, where ϕ is the space of finitely non-zero scalar sequences. As well if $X \subset Y$, then $Y_{\eta} \subset X_{\eta}$ for $\eta = \alpha$ or β. For every sequence space X, we denote $\left(X^{\delta}\right)^{\eta}$ by $X^{\delta\eta}$ where $\delta, \eta = \alpha$ or β. It is clear that $X \subset X^{\eta\eta}$ where $\eta = \alpha$ or β.

Definition 1. 1 [37,38] Let (t_i) and (s_i) be two sequences of non-negative integers satisfying

$$t_i < s_i \text{ and } \lim_{i\to\infty} s_i = +\infty.$$

The sequence $x = (x_i)$ is called to be deferred statistically convergent, if for all $\varepsilon > 0$,

$$\lim_{i\to\infty} \frac{1}{s_i - t_i} \left|\left\{j \colon t_i < j \le s_i, \left|x_j - \sigma\right| \ge \varepsilon\right\}\right| = 0,$$

holds and we shall denote it by

$$\lim_{i\to\infty} x_i = \sigma \, (\, S_d \,).$$

If $\left\{\frac{t_i}{s_i - t_i}\right\}$ is bounded, then S_d is called properly deferred.

a) The deferred statistical convergence coincides with the statistical convergence, if $s_i = i$ and $t_i = 0$, for each i.

b) If we take $s_i = k_i$ and $t_i = k_{i-1}$ (for every lacunary sequence of non-negative integers with $k_i - k_{i-1} \to \infty \; as \; i \to \infty$), and so Definition 1.1 is being turned to lacunary statistical convergence,

c) If we consider $s_i = i$ and $t_i = i - \lambda_i$, where (λ_i) is non-decreasing sequence of natural numbers such that $\lambda_1 = 1$ and $\lambda_{i+1} \leq \lambda_i+1$, then Definition 1.1 is reduced λ–statistical convergence of sequences.

Definition 1.2 [13] Let $\lambda = (\lambda_i) \in \Lambda$ and $0 < \alpha \leq 1$. The sequence (x_i) is said to be λ-statistically bounded of order α if there exists $C \geq 0$ such that

$$\lim_{i\to\infty} \frac{1}{\lambda_i^{\alpha}} \left|\left\{j \in I_i : \left|x_j\right| > C\right\}\right| = 0,$$

where $I_i = [i - \lambda_i + 1, i]$ and $\lambda_{i+1} \leq \lambda_i + 1, \lambda_1 = 1$. For the set of every λ-statistically bounded sequences of order α, we shall write $S_\lambda^\alpha(b)$. In case $\alpha = 1$, we obtain the set $S_\lambda(b)$, the set of all λ-statistically bounded sequences, in the case λ_i= i we obtain the set $S^\alpha(b)$, the set of all statistically bounded sequences of order α and also in the case $\alpha = 1, \lambda_i$= i we obtain the set $S(b)$, the set of all statistically bounded sequences.

For the spaces of all sequences bounded, convergent and null sequences $x = (x_k)$ with complex terms, we shall write $\ell_\infty, c \; and \; c_0$, respectively. These spaces are known as linear spaces of bounded, convergent and zero sequences and each one of these space is Banach space by the norm y $\|x\|_\infty = sup_k|x_k|$, where $k \in \mathbb{N} = \{1,2,3, \dots\}$, the set of positive integers.

Definition 1.3 [8] The sequence $x = (x_i)$ is called to be strongly Cesaro summable to a number σ if $lim_i \frac{1}{i}\sum_{j=1}^{i}\left|x_j - \sigma\right| = 0$. The set of strongly Cesaro summable sequences is indicated by $[C_1]$ and defined as:

$$[C_1] = \{x = (x_i) : \lim_i \frac{1}{i} \sum_{j=1}^{i} |x_j - \sigma| = 0, \qquad for\ some\ \sigma\}.$$

Definition 1.4 [12] *If* $x = (x_i)$ is a sequence such that x satisfies property $P(i)$ for all i except a set of $\alpha -$ density zero, then we say that x_i satisfies $P(i)$ *for* "almost all k according to α " and we abbreviate this by "$a.a.k(\alpha)$".

Definition 1.5 [12] Let α be any real number such that $0 < \alpha \leq 1$ and $p > 0$. A sequence $x = (x_i)$ is called to be strongly p–Cesaro summable of order α, if there is a complex number σ such that

$\lim_{i\to\infty} \frac{1}{i^a} \sum_{j=1}^{i} |x_j - \sigma|^p = 0.$

We will use w_p^α for the sequences which are strong p$-$cesaro summable of order α.

Definition 1.6 [38] A sequence $x = (x_j)$ is called to be strongly deferred p–Cesaro summable, if there is a complex number σ such that

$\lim_{i\to\infty} \frac{1}{s_i - t_i} \sum_{t_i+1}^{s_i} |x_j - \sigma|^p = 0$

then we write

$$\lim_{j\to\infty} x_j = \sigma(D_d)\ or\ \ x_j \to \sigma\ (D_d).$$

2. DEFERRED STATISTICAL CONVERGENCE

In the present part, we will give some relations between strong deferred Cesaro summability and deferred statistical convergence . It is going to show these two methods which are equivalent only for bounded sequences.

Theorem 2.1 [36] If $x_i \to \sigma\left(D_d\right), then\ x_i \to \sigma\left(S_d\right)$.

Proof. Let $x_i \to \sigma\left(D_d\right)$ and $\varepsilon > 0$ be given, then

$$\frac{1}{s_i - t_i} \sum_{j=t_i+1}^{s_i} \left|x_j - \sigma\right|$$

$$= \frac{1}{s_i - t_i}\left(\sum_{\substack{j=t_i+1 \\ |x_j-\sigma|\geq\varepsilon}}^{s_i} + \sum_{\substack{j=t_i+1 \\ |x_j-\sigma|<\varepsilon}}^{s_i}\right)\left|x_j - \sigma\right| \geq \frac{1}{s_i - t_i} \sum_{\substack{j=t_i+1 \\ |x_j-\sigma|\geq\varepsilon}}^{s_i} \left|x_j - \sigma\right|$$

$$\geq \varepsilon \frac{1}{s_i - t_i} \left|\left\{j : t_i < j \leq s_i, \left|x_j - \sigma\right| \geq \varepsilon\right\}\right| = 0,$$

and so we have the following equality

$$\lim_{i\to\infty} \frac{1}{s_i - t_i} \left|\left\{j : t_i < j \leq s_i, \left|x_j - \sigma\right| \geq \varepsilon\right\}\right| = 0.$$

Corollary 2.2 [37] If $x_i \to \sigma\,, then\ x_i \to \sigma\left(S_d\right)$.

Remark 2.3 [37] The converse of Corollary 2 .2 and Theorem 2.1aren't true, in general.

To show this, we choose a sequence $x = (x_i)$ defined by

$$x_j = \begin{cases} j^2, & \left[\left|\sqrt{s_i}\right|\right] - l_0 < j \leq \left[\left|\sqrt{s_i}\right|\right], \quad i = 1,2,3, \dots, \\ 0, & otherwise, \end{cases}$$

where $(s_i\ )$ is a monotone increasing sequence and $l_0 \neq 0$ is an arbitrary fixed natural number.

Now we consider a sequence $(t_i\ )$ satisfying the following condition

$$0 < t_i \leq [|\sqrt{s_i}|] - l_0,$$

then for any $\varepsilon > 0$ we can write

$$\lim_{i\to\infty} \frac{1}{s_i - t_i}\ \left|\{j: t_i < j \leq s_i\ \ |x_j - 0| \geq \varepsilon\}\right|\ = \frac{l_0}{s_i - t_i} \to 0,$$

$i.e.,\ x_i \to 0\ (S_d)$ and on the other hand we have the following inequality

$$\frac{1}{s_i-t_i}\sum_{j=t_i+1}^{s_i}|x_j - 0| \geq \frac{l_0([|\sqrt{s_i}|]-l_0)^2}{s_i-t_i} \to l_0,\ \ i \to \infty$$

so $x = (x_i)$ is not D_d convergent to zero. It's clear the sequence doesn't converge to zero in usual case.

Theorem 2.4 [37] If $x \in l_\infty$ *and* $x_i \to \sigma\ (S_d)$ *then* $x_i \to \sigma\ (D_d)$.

Proof. Suppose that $x \in l_\infty$ *and* $x_i \to \sigma\ (S_d)$. Then there exists a positive real number C such that $|x_i - \sigma| \leq C$ holds for all n.

So, the inequality

$$\frac{1}{s_i - t_i}\sum_{j=t_i+1}^{s_i}|x_j - \sigma| = \frac{1}{s_i - t_i}\left(\sum_{\substack{j=t_i+1 \\ |x_i-\sigma|\geq\varepsilon}}^{s_i}|x_j - \sigma| + \sum_{\substack{j=t_i+1 \\ |x_i-\sigma|<\varepsilon}}^{s_i}|x_j - \sigma|\right)$$

$$\leq\ \frac{1}{s_i - t_i}\left(C\sum_{\substack{j=t_i+1 \\ |x_i-\sigma|\geq\varepsilon}}^{s_i} 1\ + \varepsilon\sum_{\substack{j=t_i+1 \\ |x_i-\sigma|<\varepsilon}}^{s_i} 1\right)$$

$$\leq C\ \frac{1}{s_i - t_i}\ \left|\{j: t_i < j \leq s_i, |x_j - \sigma| \geq \varepsilon\}\right|$$

$$+\varepsilon\ \frac{1}{s_i-t_i}\ \ \left|\{j: t_i < j \leq s_i, |x_j - \sigma| < \varepsilon\}\right|\ ,$$

is hold. From the limit relation that we get

$$\lim_{n\to\infty} \frac{1}{s_i - t_i} \sum_{j=t_i+1}^{s_i} |x_j - \sigma| = 0 .$$

So, we completed the proof.

In the following theorems, we compare statistical convergence and deferred statistical convergence under several restrictions on (s_i) or (t_i).

Theorem 2.5 [37] If the sequence $\left\{\frac{t_i}{s_i - t_i}\right\}_{i\in\mathbb{N}}$ is bounded, then $x_i \to \sigma\ (S)$ implies

$x_i \to \sigma\ (S_d)$.

Proof. Let $x_i \to \sigma\ (S)$, then the relation

$$\lim_{i\to\infty} \frac{1}{i} \left|\left\{j: 0 < j \le i, |x_j - \sigma| \ge \varepsilon\right\}\right| = 0$$

holds for all $\varepsilon > 0$. Then the sequence

$$\left\{\frac{\left|\left\{j: 0 < j \le s_i, |x_j - \sigma| \ge \varepsilon\right\}\right|}{s_i}\right\}$$

is convergent to 0. So the relation

$$\left\{j: t_i < j \le s_i, |x_j - \sigma| \ge \varepsilon\right\} \subset \left\{j: 0 < j \le s_i, |x_j - \sigma| \ge \varepsilon\right\}$$

holds and also the inequality

$$\left|\left\{j: t_i < j \le s_i, |x_j - \sigma| \ge \varepsilon\right\}\right| \le \left|\left\{j: 0 < j \le s_i, |x_j - \sigma| \ge \varepsilon\right\}\right|$$

holds. From the last inequality we have

$$\frac{1}{s_i - t_i} \left|\left\{j: t_i < j \le s_i, |x_j - \sigma| \ge \varepsilon\right\}\right|$$

$$\le \left(1 + \frac{t_i}{s_i - t_i}\right) . \frac{1}{s_i} \left|\left\{j: 0 < j \le s_i, |x_j - \sigma| \ge \varepsilon\right\}\right|,$$

and form the limit relation we gain

$$x_i \to \sigma(S_d).$$

So, desired result is obtained.

Corollary 2.6 [37] Let (s_i) be an arbitrary sequence with $s_i < i$ for all $i \in \mathbb{N}$ and $\left\{\frac{i}{s_i - t_i}\right\}_{i\in\mathbb{N}}$ be a bounded sequence. Then $x_i \to \sigma\,(S)$ implies $x_i \to \sigma(S_d)$.

Remark 2.7 [37] The converse of Theorem 2.5 isn't true even if $\left\{\frac{t_i}{s_i - t_i}\right\}_{i\in\mathbb{N}}$ is bounded.

Example 2.8 [37] Let us consider $t_i = 2i, s_i = 4i\,, i \in \mathbb{N}\; and\; define\; a\; sequence\; x = (x_i)\; by$

$$x_i = \begin{cases} \dfrac{i+1}{2}, & i\ is\ odd, \\ -\dfrac{i}{2}, & i\ is\ even. \end{cases}$$

It is clear that $x_i \to 0(D_d)$ and from Theorem 2.1 we get $x_i \to 0(S_d)$, but

$$\lim_{i\to\infty} \frac{1}{i}\ \left|\left\{j: 0 < j \le i, \left|x_j - 0\right| \ge \varepsilon\right\}\right| \ne 0$$

Remark 2.9 [37] Two properly deferred statistically convergent method shouldn't include any other. Let

$$x_i = \begin{cases} j+1, & i = 2j+1, \\ -j, & i = 2j. \end{cases}$$

It is clear that $x_i \to 0\left(S_d\right)$, for $t_i = 2i, s_i = 4i$, but $x_i \to \frac{1}{2}\left(S_d\right)$, for $t_i = 2i - 1$, $s_i = 4i - 1$, for all $i \in \mathbb{N}$.

Theorem 2.10 [37] Let $s_i = i$ for all $i \in \mathbb{N}$. Then, $x_i \to \sigma(S_d)$ if and only if $x_i \to \sigma(S)$

Proof.($\Rightarrow$) Let us assume that $x_i \to \sigma(S_d)$. Then for any $i \in \mathbb{N}$,

$$t_i = i^{(1)} > t_{i^{(1)}} = i^{(2)} > t_{i^{(2)}} = i^{(3)} > \cdots,$$

and we may write the set $\left\{j \le i: \left|x_j - \sigma\right| \ge \varepsilon\right\}$ as

$$\left\{j \le i: \left|x_j - \sigma\right| \ge \varepsilon\right\} = \left\{j \le i^{(1)}: \left|x_j - \sigma\right| \ge \varepsilon\right\} \cup \left\{i^{(1)} < j \le i: \left|x_j - \sigma\right| \ge \varepsilon\right\},$$

the set $\left\{1 < j \le i^{(1)}:\ \left|x_j - \sigma\right| \ge \varepsilon\right\}$ *as*

$$\left\{1 < j \le i^{(1)}:\ \left|x_j - \sigma\right| \ge \varepsilon\right\}$$

$$= \left\{j \le i^{(2)}: \left|x_j - \sigma\right| \ge \varepsilon\right\} \cup \left\{i^{(2)} < j \le i^{(1)}: \left|x_j - \sigma\right| \ge \varepsilon\right\},$$

the set $\{j \le i^{(2)} : \ |x_j - \sigma| \ge \varepsilon\}$ *as*

$$\{j \le i^{(2)}: |x_j - \sigma| \ge \varepsilon\} = \{j \le i^{(3)}: |x_j - \sigma| \ge \varepsilon\} \cup \{i^{(3)} < j \le i^{(2)}: |x_j - \sigma| \ge \varepsilon\},$$

and if this process is continued we achieve

$$\{j \le i^{(l-1)} : \ |x_j - \sigma| \ge \varepsilon\}$$

$$= \{j \le i^{(l)}: |x_j - \sigma| \ge \varepsilon\} \cup \{i^{(l)} < j \le j^{(l-1)}: |x_j - \sigma| \ge \varepsilon\}.$$

For a certain positive integer $l > 0$ depending on i such that $i^{(l)} \ge 1 \ and \ i^{(l+1)} = 0$. From the above discussion, the relation

$$\frac{1}{i}|\{j \le i: |x_i - \sigma| \ge \varepsilon\}|$$

$$= \sum_{r=0}^{l} \frac{i^{(r)} - i^{(r+1)}}{i} \frac{1}{i^{(r)} - i^{(r+1)}} |\{i^{(r+1)} < j \le i^{(r)}: |x_j - \sigma| \ge \varepsilon\}|.$$

Holds for any i. This relation gives that statistical convergence of the sequence (x_i) to σ is a linear combination of following sequence

$$\frac{1}{i^{(r)} - i^{(r+1)}}\{|i^{(r+1)} < j \le i^{(r)}: |x_j - \sigma| \ge \varepsilon|\}_{r\in\mathbb{N}}$$

Let us consider the matrix

$$b_{i,r} = \begin{cases} \dfrac{i^{(r)} - i^{(r+1)}}{i}, & r = 0,1,2,3,\dots,l, \\ 0, & otherwise, \end{cases}$$

where $i^{(0)} = i$.

The matrix $(b_{i,r})$ is satisfied the Silverman Toeplitz theorem . So we have

$$\lim_{i\to\infty} \frac{1}{i}|\{j \le i: |x_j - \sigma| \ge \varepsilon\}| = 0$$

since

$$\frac{1}{i^{(r)} - i^{(r+1)}}|\{i^{(r+1)} < j \le i^{(r)}: |x_j - \sigma| \ge \varepsilon\}| \to 0, i \to \infty.$$

($\Leftarrow$) Since $s_i = i$ is satisfied (1.2), and so the inverse of the theorem is a simple consequence of Theorem 2.5.

Corollary 2.11 [37] Assume that (s_i) consists of all positive integers and so,

$x_i \to \sigma(S_d)$ implies $x_i \to \sigma(S)$.

Now we compare the different S_d methods under the next restriction $k_i \to \infty$

$$t_i \leq r_i < k_i \leq s_i, \text{ for all } i \in \mathbb{N} \qquad (2.1)$$

Theorem 2.12 [37] Let (r_i) and (k_i) be two sequences of positive natural numbers satisfying (2.1) such that the sets

$$\{j: t_i < j \leq r_i\} \; and \; \{j: k_i < j \leq s_i\}$$

are finite sets for all $i \in \mathbb{N}$. Then $x_i \to \sigma(S_d')$) implies $x_i \to \sigma(S_d)$.

Proof. Let's consider the sequence $x = (x_i)$ such that $x_i \to \sigma(S_d)$. For an arbitrary $\varepsilon > 0$, the equality

$$\{j: t_i < j \leq s_i, |x_j - \sigma| \geq \varepsilon\} = \{j: t_i < j \leq r_i, |x_j - \sigma| \geq \varepsilon\}$$

$$\cup \{j: r_i < j \leq k_i, |x_j - \sigma| \geq \varepsilon\} \cup \{j: k_i < j \leq s_i, |x_j - \sigma| \geq \varepsilon\}$$

and the inequality

$$\frac{1}{s_i - t_i} |\{j: t_i < j \leq s_i, |x_j - \sigma| \geq \varepsilon\}|$$

$$\leq \frac{1}{k_i - r_i} \left|\left\{j: t_i < j \leq r_i, \quad \left||x_j - \sigma|\right| \geq \varepsilon\right\}\right|$$

$$+ \frac{1}{k_i - r_i} |\{j: r_i < j \leq k_i, \; |x_j - \sigma| \geq \varepsilon\}|$$

$$+ \frac{1}{k_i - r_i} |\{j: k_i < j \leq s_i, \; |x_j - \sigma| \geq \varepsilon\}|$$

are hold.

On taking limits when $i \to \infty$, we get

$$\lim_{n \to \infty} \frac{1}{s_i - t_i} |\{j: t_i < j \leq s_i, |x_j - \sigma| \geq \varepsilon\}| = 0.$$

This proves our assertion.

Theorem 2.13 [37] Let (t_i), (r_i), (k_i), and (s_i) be sequences of positive natural numbers satisfied (2.1) such that

$$\lim_{i\to\infty}\frac{s_i - t_i}{k_i - r_i} = b > 0.$$

Then, $x_i \to \sigma\big(S_d\ \big)\ implise\ \ x_i \to \sigma(S_d')$

Proof. It's easy to see the inclusion

$$\left\{j\colon r_i + 1 \le j \le k_i, \left|x_j - \sigma\right| \ge \varepsilon\right\} \subset \left\{j\colon t_i + 1 \le j \le s_i, \left|x_j - \sigma\right| \ge \varepsilon\right\}$$

and the inequality

$$\left\{j\colon r_i + 1 \le j \le r_i, \left|x_j - \sigma\right| \ge \varepsilon\right\} \le \left\{j\colon t_i + 1 \le j \le s_i, \left|x_j - \sigma\right| \ge \varepsilon\right\}$$

are true. So, we have

$$\frac{1}{k_i - r_i}\ \left|\left\{j\colon r_i + 1 < j \le k_i,\ \ \left|x_j - \sigma\right| \ge \varepsilon\right\}\right|$$

$$\le \frac{s_i - t_i}{k_i - r_i}\ \frac{1}{s_i - t_i}\ \left|\left\{j\colon t_i + 1 < j \le s_i,\ \ \left|x_j - \sigma\right| \ge \varepsilon\right\}\right|.$$

Taking limits as $i \to \infty$ the desired result is obtained.

3. STATISTICAL CONVERGENCE OF ORDER α

In this part we give the concepts of statistical convergence of order α and strong p-Cesaro summability of order α for sequences of complex or real numbers. Furthermore some relations between the statistical convergence of order α and strong p-Cesaro summability of order α are given.

Lemma 3.1 [12] Let $E \subseteq \mathbb{N}$. Then, $\delta_\beta(E) \leq \delta_\alpha(E)$ if $0 < \alpha \leq \beta \leq 1$

Proof. Let $0 < \alpha \leq \beta \leq 1$. Since $n^\alpha \leq n^\beta$ so that $\frac{1}{n^\beta} \leq \frac{1}{n^\alpha}$ for each $n \in \mathbb{N}$, we have

$$\frac{1}{n^\beta}|\{k \leq n: k \in E\}| \leq \frac{1}{n^\alpha}|\{k \leq n: k \in E\}|$$

From this inequality we have $\delta_\beta(E) \leq \delta_\alpha(E)$.

From Lemma 3.1 we have that if E has zero $\alpha-$density for some $0 < \alpha \leq 1$, then it has zero natural density.

Theorem 3.2 [12] Let $0 < \alpha \leq 1$ and $x = (x_k)$, $y = (y_k)$ be sequences of complex number.

a) If $S^\alpha\text{–}\lim x_k = x_0$ and $c \in \mathbb{C}$, then $S^\alpha\text{–}\lim cx_k = cx_0$

b) If $S^\alpha\text{–}\lim x_k = x_0$ and $S^\alpha\text{–}\lim y_k = y_0$ then $S^\alpha\text{–}\lim(x_k + y_k) = x_0 + y_0$.

Proof. (a) follows from

$$\frac{1}{n^\alpha}|\{k \leq n: |cx_k - cx_0| \geq \varepsilon\}| \leq \frac{1}{n^\alpha}\left|\{k \leq n: |x_k - x_0| \geq \frac{\varepsilon}{|c|}\}\right|$$

and that of (b) follows from

$$\frac{1}{n^\alpha}|\{k \leq n: |(x_k + y_k - (x_0 + y_0)| \geq \varepsilon\}| \leq \frac{1}{n^\alpha}\left|\left\{k \leq n: |x_k - x_0| \geq \frac{\varepsilon}{2}\right\}\right|$$

$$+\frac{1}{n^{\alpha}}\left|\{k \leq n: |y_k - y_0| \geq \frac{\varepsilon}{2}\}\right|$$

Note that S^{α} is different from S_{λ} defined in [9], in general. If we take $\lambda_n = n^{\alpha}$ for $0 < \alpha \leq 1$, then $S^{\alpha} \subseteq S_{\lambda}$. If $\lambda_n = n^{\alpha}$ with $\alpha = 1$, that is $\lambda_n = n$ then $S^{\alpha} = S_{\lambda} = S$ [9].

$c \subset S^{\alpha}$ for every $0 < \alpha \leq 1$, but the converse doesn't hold. For instance the sequence $x = (x_k)$ defined by

$$x_k = \begin{cases} 1, & k = n^3, \\ 0, & k \neq n^3 \end{cases} \tag{3.1}$$

$x \in S^{\alpha}$, for $\alpha > \frac{1}{3}$, but it isn't convergent.

Theorem 3.3 [12] Let $0 < \alpha \leq 1$. Then the inclusion $S^{\alpha} \subseteq S^{\beta}$ is strict for some $\alpha < \beta$.

Proof. If $0 < \alpha \leq \beta \leq 1$, then

$$\frac{1}{n^{\beta}}|\{k \leq n: |x_k - l| \geq \varepsilon\}| \leq \frac{1}{n^{\alpha}}|\{k \leq n: |x_k - l| \geq \varepsilon\}|$$

for every $\varepsilon > 0$ and this allows that $S^{\alpha} \subseteq S^{\beta}$. Consider a sequence x defined by

$$x_k = \begin{cases} 1, & k = n^2, \\ 0, & k \neq n^2. \end{cases} \tag{3.2}$$

Then $x \in S^{\beta}$ for $\frac{1}{2} < \beta \leq 1$, but $x \notin S^{\alpha}$ for $0 < \alpha \leq \frac{1}{2}$ so the inclusion is strict.

Corollary 3.4 [12]

a) $S^{\alpha} = S^{\beta}$ if and only if $\alpha = \beta$.

b) $S^{\alpha} = S$ if and only if $\alpha = 1$.

Theorem 3.5 [12] Let $0 < \alpha \leq \beta \leq 1$ and $x = (x_k)$ be a statistically convergent sequence of order α such that $S^{\alpha} - \lim x_k = l$. Then there is a subsequence $y = (y_k)$ of $x = (x_k)$ such that $\lim y_k = l$.

Theorem 3.6 [12] Let $0 < \alpha \leq \beta$ and $p > 0$. Then, $w_p^\alpha \subseteq w_p^\beta$ and the inclusion is strict for $\alpha < \beta$.

Proof. Let $x = (x_k) \in w_p^\alpha$. Then, given α and β such that $0 < \alpha \leq \beta \leq 1$ and a positive real number p we may write

$$\frac{1}{n^\beta}\sum_{k=1}^{n}|x_k - l|^p \leq \frac{1}{n^\alpha}\sum_{k=1}^{n}|x_k - l|^p$$

and this given that $w_p^\alpha \subseteq w_p^\beta$.

To indicate that the inclusion is strict, consider the sequence $x = x_k$ defined in (3.2). It is easy to see that

$$\frac{1}{n^\beta}\sum_{k=1}^{n}|x_k - 0|^p \leq \frac{\sqrt{n}}{n^\beta} = \frac{1}{n^{\beta-\frac{1}{2}}} .$$

Since $\frac{1}{n^{\beta-\frac{1}{2}}} \to 0$ as $n \to \infty$, then $w_p^\beta - \lim x_k = 0$. $i.e.$ $x \in w_p^\beta$ for $\frac{1}{2} < \beta \leq 1$, but since

$$\frac{\sqrt{n}-1}{n^\alpha} \leq \frac{1}{n^\alpha}\sum_{k=1}^{n}|x_k - 0|^p$$

and $\frac{\sqrt{n}-1}{n^\alpha} \to 0\ as\ n \to \infty$, then $x \notin w_p^\alpha$ $0 < \alpha \leq \frac{1}{2}$. This completes the proof.

Corollary 3.7 [12] Let $0 < \alpha \leq \beta$ and $0 < p < \infty$. Then

a) $w_p^\alpha = w_p^\beta$ if and only if $\alpha = \beta$.

b) $w_p^\alpha \subseteq w_p$, $\alpha \in (0,1]$.

Theorem 3.8 [12] Let $0 < \alpha$ and $0 < p < q < \infty$. Then, $w_q^\alpha \subset w_p^\alpha$.

Taking $\alpha = 1$ in Theorem 3.8 we deduce the following result:

If $0 < p < q < \infty$, then $w_q \subset w_p$.

Theorem 3.9 [12] Let $0 < \alpha \leq \beta \leq 1$ and $0 < p < \infty$, then $w_p^\alpha \subset S^\beta$

Proof. Let $\varepsilon > 0$, we have

$$\sum_{k=1}^{n} |x_k - l|^p \geq |\{k \leq n : |x_k - l|^p \geq \varepsilon\}|.\varepsilon^p$$

and so that

$$\frac{1}{n^\alpha}\sum_{k=1}^{n} |x_k - l|^p \geq \frac{1}{n^\alpha}|\{k \leq n : |x_k - l|^p \geq \varepsilon\}|.\varepsilon^p \geq \frac{1}{n^\beta}|\{k \leq n : |x_k - l|^p \geq \varepsilon\}|.\varepsilon^p$$

Hence we get $w_p^\alpha \subset S^\beta$.

Taking $\beta = \alpha$ in theorem 3.9, the next result is obtained.

Corollary 3.10 [12] Let $0 < \alpha \leq 1$ and $0 < p < \infty$ then $w_p^\alpha \subset S^\alpha$

Remark Note that the converse of theorem 3.9 doesn't hold, in general. The sequence

$$x_k = \begin{cases} \frac{1}{\sqrt{k}}, & k \neq m^3, \\ 1, & k = m^3. \end{cases}$$

is an example for this case. It is clear that $x \in l_\infty$ and $x \in S^\alpha$ for each α $\left(\frac{1}{3} < \alpha \leq 1\right)$. First recall that the inequality

$$\sum_{k=1}^{n} \frac{1}{\sqrt{k}} > \sqrt{n}$$

holds for every positive integer $n \geq 2$. Define $H_n = \{k \leq n: k \neq m^3, m = 1,2,3, \ldots\}$ and take $\mathrm{p} = 1$. Since

$$\sum_{k=1}^{n} |x_k|^p = \sum_{k=1}^{n} |x_k| = \sum_{k \in H_n, 1 \leq k \leq n}^{n} |x_k| + \sum_{k \notin H_n, 1 \leq k \leq n}^{n} |x_k|$$

$$\sum_{k \in H_n, 1 \leq k \leq n}^{n} \frac{1}{\sqrt{k}} + \sum_{k \notin H_n, 1 \leq k \leq n}^{n} 1 > \sum_{k=1}^{n} \frac{1}{\sqrt{k}} > \sqrt{n}$$

we have

$$\frac{1}{n^\alpha} \sum_{k=1}^{n} |x_k|^p = \frac{1}{n^\alpha} \sum_{k=1}^{n} |x_k| > \frac{1}{n^\alpha} \sum_{k=1}^{n} \frac{1}{\sqrt{k}} > \frac{1}{n^\alpha} \sqrt{n} = \frac{1}{n^{\alpha - \frac{1}{2}}} \to \infty \; as \; n \to \infty$$

and so that $x \notin w_p^\alpha \; for \; 0 < \alpha < \frac{1}{2} \; if \; p = 1$. Therefore $x \in s^\alpha - w_p^\alpha \; for \; \frac{1}{3} < \alpha < \frac{1}{2} \; if \; p = 1$.

Corollary 3.11 [12] Let $0 < \alpha \leq 1$ and $p > 0$. Then $w_p^\alpha \subset S$. The inclusion is strict if $0 < \alpha < 1$.

Proof. The inclusion part of the proof is easy. For strictness consider the sequence $x = (x_k)$ defined in (3.1). Then, clearly $S\text{–}\,lim\, x_k = 0$, i.e. $x \in S \; but \; x \notin w_p^\alpha$ for $0 < \alpha \leq \frac{1}{3} \; and \; p = 1$. Indeed it is easy to see that

$$\frac{1}{n^\alpha} \sum_{k=1}^{n} |x_k - 0|^p = \frac{1}{n^\alpha} \sum_{k=1}^{n} |x_k| \geq \frac{\sqrt[3]{n} - 1}{n^\alpha}.$$

Since $\frac{\sqrt[3]{n}-1}{n^\alpha} \to \infty$ $(\frac{\sqrt[3]{n}-1}{n^\alpha} \to 1 \; if \; \alpha = \frac{1}{3})$ as $n \to \infty$, then $x \notin w_p^\alpha$ for $0 < \alpha \leq \frac{1}{3}$ and $p = 1$. Conversely $x \in S\text{–}\,w_p^\alpha \; for \; 0 < \alpha \leq \frac{1}{3}$ and $p = 1$. This completes the proof.

4. ON SOME GENERALIZATION OF STATISTICAL BOUNDEDNESS

In this part we introduce the concept of λ-statistical bounded of order α and give some relations between λ-statistically bounded sequences of order α and statistically bounded sequences.

Let's start by giving an example of our results.

A bounded sequence is statistically bounded but, the converse isn't true, as the next example shows.

Example 4.1 [13] Define a sequence $x = (x_k)$ such as

$$x_k = \begin{cases} k, & if\ k\ is\ a\ square; \\ 0, & if\ k\ is\ not\ a\ square. \end{cases}$$

(x_k) isn't bounded sequence. However, $\delta\left(\left\{k: |x_k| > \frac{1}{2}\right\}\right) = 0$ and so $x = (x_k)$ is statistically bounded.

Proposition 4.2 [13] A convergent sequence is statistically bounded.

Proof. Omitted.

Proposition 4.3 [13] A statistically convergent sequence is statistically bounded, but the converse does not hold.

Proof. If $x = (x_k)$ is statistically convergent to L, then for any $\varepsilon > 0$, we can write $\delta(\{k: |x_k - L| > \varepsilon\}) = 0$. So $\{k: |x_k| > |L| + \varepsilon\} \subset \{k: |x_k - L| > \varepsilon\}$ and $|x_k| \leq |L| + \varepsilon\ a.a.k.$

Proposition 4.4 [13] If $st - lim_k x_k = 0$ and $(y_k) \in S(b)$, then $(x_k y_k)$ converges statistically to 0.

Theorem 4.5 [13] $(x_k) \in S(b)$ $\Leftrightarrow$ there exists a bounded sequence $y = (y_k)$ such that $x_k = y_k\ a.a.k.$

Proof. Let $x \in S(b)$ and define a sequence $y = (y_k)$ such that

$$y_k = \begin{cases} x_k & if\ k \in A, \\ 0 & otherwise. \end{cases}$$

Then $y = (y_k) \in l_\infty$ and $y_k = x_k\ a.a.k.$

Conversely, as $y = (y_k) \in l_\infty$ so there exists $L \geq 0$ such that$|\ y_k| \leq L\ for\ all\ k \in \mathbb{N}.$ *Let* $D = \{k \in \mathbb{N}: x_k \neq y_k\}$. Clearly $\delta(D) = 0$. This gives $|x_k| \leq L\ a.a.k$ because $\{k \in \mathbb{N}: |x_k| > L\} \subset D$.

Theorem 4.6 [13] A statistically Cauchy sequence is statistically bounded, but the converse doesn't need to be true.

Proof. If x is a statistically Cauchy sequence, then there exists a number $N = N(\varepsilon)$ such that $|x_k - x_N| < \varepsilon\ a.a.k.$ Hence $|x_k| < L$ a.a.k, where $= \varepsilon + |x_N|$.

The sequence $x = (x_k) = (-2,2,-2,2, \dots) \in S(b)$, but isn't statistically Cauchy.

Theorem 4.7 [13] a) $S(b)$ $i$sn't symmetric,

$b)$ $S(b)$ Is normal,

$c)$ $S(b)$ Is monotone

$d)$ $S(b)$ Is a sequence algebra.

Proof. Omitted.

Proposition 4.8 [13] $S(b)^\alpha = S(b)^\beta = \phi$.

Proof: To indicate that $S(b)^\alpha = \phi$, it is sufficient to show that $S(b)^\alpha \subset \phi$ since $\phi \subset S(b)^\alpha$ is obviousl. Let $(a_k) \in S(b)^\alpha$. Then $\sum_k |a_k x_k| < \infty\ for\ all\ x = (x_k) \in S(b)$. Let $(a_k) \notin \phi$, and choose a sequence (k_i) such that $k_i > i^2$ and $a_{k_i} \neq 0$. Define $x = \{x_k\}\ as$

$$x_{k_i} = \begin{cases} \dfrac{1}{a_{k_i}}, & i \in \mathbb{N}, \\ 0, & otherwise. \end{cases}$$

Then $(x_k) \in S(b)$ and $\sum_k |a_k x_k| = \infty$, and so $(a_k) \in S(b)^\alpha$.

Theorem 4.9 [13] $l_\infty \subset S^\alpha(b)$ for $0 < \alpha \leq 1$.

Proof. The inclusion part of the proof is easy. For strictness, consider a sequence $x = (x_k)$ defined by

$$x_k = \begin{cases} k, & k = n^2, \\ 0, & k \neq n^2. \end{cases} \quad k \in \mathbb{N} \tag{4.1}$$

Then $(x_k) \notin l_\infty$. But $\lim_{n\to\infty} \frac{1}{n^\alpha} |\{k \leq n: |x_k| > 0\}| = 0,\ for\ \alpha > \frac{1}{2}\ and\ so\ (x_k) \in S^\alpha(b)$, $for\ \alpha > \frac{1}{2}$.

Theorem 4.10 [13] Let $0 < \alpha \leq \beta \leq 1$ then $S^\alpha(b) \subset S^\beta(b)$, and the inclusion is strict.

Proof. $If\ \ 0 < \alpha \leq \beta \leq 1,$ then

$$\lim_{n\to\infty} \frac{1}{n^\beta} |\{k \leq n: |x_k| > M\}| \leq \lim_{n\to\infty} \frac{1}{n^\alpha} |\{k \leq n: |x_k| > M\}|,$$

and this implies $S^\alpha(b) = S^\beta(b)$ this inclusion may be strict for the stricness, consider the sequence $x = (x_k)$ defined by (4.1), then $(x_k) \in S^\beta(b)\ for\ \frac{1}{2} < \beta \leq 1,$ but $(x_k) \notin S^\alpha(b)\ for\ 0 < \alpha \leq \frac{1}{2}$.

Corollary 4.11 [13] Let $0 < \alpha \leq \beta \leq 1, then$ a) $S^\alpha(b) = S^\beta(b)\ \ if\ and\ only\ if\ \alpha = \beta$.

b) $[S(b)]^\alpha \subset S(b)$

Theorem 4.12 [13] $For\ 0 < \alpha \leq 1,\ S^\alpha \subset S^\alpha(b)$.

Proof. Let $(x_k) \in S^\alpha$. Then there exists some $L \in \mathbb{C}$, such that

$$\lim_{n\to\infty} \frac{1}{n^\alpha} |\{k \leq n: |x_k - L| \geq \varepsilon\}| = 0, \text{ for every } \varepsilon > 0$$

and so

$$\lim_{n\to\infty}\frac{1}{n^{\alpha}}|\{k\le n: |x_k|>|L|+\varepsilon\}|\le \lim_{n\to\infty}\frac{1}{n^{\alpha}}|\{k\le n: |x_k-L|\ge\varepsilon\}|.$$

Theorem 4.13 [13] For each $\lambda\in\Lambda,\ l_{\infty}\subset S_{\lambda}(b)$.

Proof. The inclusion part of the proof is easy. For strict inclusion define a sequence x by

$$x_k=\begin{cases}k, & k=n^2,\\ 0, & k\ne n^2\ for\ k\in\mathbb{N}\end{cases}$$

and for $\lambda=(n)$.

Theorem 4.14 [13] *Let* $\lambda,\mu\in\Lambda$. Then

a) If $\liminf_{n\to\infty}\frac{\lambda_n}{\mu_n}>0$, then $S_{\mu}(b)\subset S_{\lambda}(b)$.

b) If $\lim_{n\to\infty}\frac{\lambda_n}{\mu_n}=1$, then $S_{\lambda}(b)\subset S_{\mu}(b)$.

c) If $\lim_{n\to\infty}\frac{\lambda_n}{\mu_n}=1, then\ S_{\lambda}(b)=S_{\mu}(b)$.

Proof. **(a)** Let $\liminf_{n\to\infty}\frac{\lambda_n}{\mu_n}>0\ and\ (x_k)\in S_{\mu}(b)$.Then

$$\lim_{n\to\infty}\frac{1}{\mu_n}\left|\left\{k\in\acute{l}_n: |x_k|>M\right\}\right|=0,$$

where $\acute{l}_n=[n-\mu_n+1,n]$. *Now* $\left\{k\in\acute{l}_n: |x_k|>M\right\}\supset\{k\in l_n: |x_k|>M\}$

and *so* $\left|\frac{1}{\mu_n}\left\{k\in\acute{l}_n: |x_k|>M\right\}\right|\ge\frac{\lambda_n}{\mu_n}\frac{1}{\lambda_n}|\{k\in l_n: |x_k|>M\}|$ which in turn implies that

$$\lim_{n\to\infty}\frac{1}{\lambda_n}|\{k\in l_n: |x_k|>M\}|=0.$$

b) *Let* $(x_k)\in S_{\lambda}(b),\ then$

$$\frac{1}{\mu_n}\left\{k\in\acute{l}_n: |x_k|>M\right\}=\frac{1}{\mu_n}\left\{k\in\acute{l}_n-l_n: |x_k|>M\right\}+\frac{1}{\mu_n}\{k\in l_n: |x_k|>M\}$$

$$\leq \frac{1}{\mu_n}(\mu_n - \lambda_n) + \frac{1}{\lambda_n}|\{k \in I_n: |x_k| > M\}|$$

$$= \left(1 - \frac{\lambda_n}{\mu_n}\right) + \frac{1}{\lambda_n}|\{k \in I_n: |x_k| > M\}| \rightarrow 0 \;\; as\; n \rightarrow \infty$$

and so $(x_k) \in S_\mu(b)$

c) Proof follows from (a) and (b).

Corollary 4.15 [13] Let $\lambda = (\lambda_n) \in \Lambda$.

a) lim $\inf_{n\to\infty} \frac{\lambda_n}{n} > 0$ impiles $S(b) \subset S_\lambda(b)$,

b) *If* $\lim_{n\to\infty} \frac{\lambda_n}{n} = 1, then\; S_\lambda(b) \subset S(b)\; and\; hence\; S_\lambda(b) = S(b)$.

Theorem 4.16 $l_\infty \subset S_\lambda^\alpha(b)$, for each $0 < \alpha \leq 1$ and $\lambda \in \Lambda$

Proof. Omitted.

Theorem 4.17 [13] Let $\lambda = (\lambda_n)$ and $\mu = (\mu_n)$ be two sequences in Λ such that $\lambda_n \leq \mu_n$ and $0 < \alpha \leq \beta \leq 1$.

a) If lim $\inf_{n\to\infty} \frac{\lambda_n^\alpha}{\mu_n^\beta} > 0, then\; S_\lambda^\beta(b) \subset S_\lambda^\alpha(b)$.

b) If $\lim_{n\to\infty} \frac{\lambda_n^\alpha}{\mu_n^\beta} = 1$, then $S_\lambda^\alpha(b) \subset S_\lambda^\beta(b)$.

Proof is easy so omitted.

Corollary 4.18 [13] Let $\lambda \in \Gamma$ and $0 < \alpha \leq 1$.

a) If lim $\inf_{n\to\infty} \frac{\lambda_n^\alpha}{n} > 0, then\; S(b) \subset S_\lambda^\alpha(b)$.

b) If $\lim_{n\to\infty} \frac{\lambda_n^\alpha}{n} = 1, then\; S_\lambda^\alpha(b) \subset S(b)$ and so that $S_\lambda^\alpha(b) = S(b)$.

5. DEFERRED STATISTICAL CONVERGENCE OF ORDER α

In this part we introduce the concepts of deferred statistical convergence sequences of order α and strong p −deferred Cesàro summability sequences of order α for sequence and give some relations between deferred statistically convergent sequences of order α and strongly p −deferred Cesàro summable sequences of order α.

Definition 5.1 Let (t_i) and (s_i) be two sequences of non-negative integers satisfying

$t_i < s_i$ and $\lim_{i\to\infty} s_i = +\infty$,

and $\alpha \in (0,1]$ be given. The sequence $x = (x_j)$ is called to be deferred statistically convergent of order α to σ if, for every $\varepsilon > 0$,

$$\lim_{i\to\infty} \frac{1}{(s_i - t_i)^\alpha} \left|\left\{j: t_i < j \le s_i, \left|x_j - \sigma\right| \ge \varepsilon\right\}\right| = 0, \qquad \textbf{(5.1)}$$

In that case we write $S_d^\alpha - lim\, x_j = \sigma$ or $x_i \to \sigma\ (S_d^\alpha)$. The set of all deferred statistically convergent sequences of order α have been denoted by S_d^α. If $s_i = i$ and $t_i = 0$, then $S_d^\alpha = S^\alpha$ and also in the special case $s_i = i$ and $t_i = 0$ and $\alpha = 1$, then $S_d^\alpha = S$.

The deferred statistical convergence of order α isn't well defined for $\alpha > 0$. For this let (x_j) be defined as:

$$x_j = \begin{cases} 1, & j = 2i \\ 0, & j \ne 2i \end{cases}, \quad i \in \mathbb{N}.$$

Then, both

$$\lim_{i\to\infty} \frac{1}{i^\alpha} \left|\left\{t_i < j \le s_i : \left|x_j - 1\right| \ge \varepsilon\right\}\right| \le \lim_{i\to\infty} \frac{i}{2i^\alpha} = 0$$

and

$$\lim_{i\to\infty} \frac{1}{i^\alpha} \left|\left\{t_i < j \le s_i : \left|x_j - 1\right| \ge \varepsilon\right\}\right| \le \lim_{i\to\infty} \frac{i+1}{2i^\alpha} = 0$$

for $\alpha > 1$ and $s_i = 4i, t_i = 3i$. So $x = (x_j)$ is deferred statistically convergent of order α, both to 1 and 0, it means $S_d^\alpha - lim\, x_j = 1$ or $S_d^\alpha - lim\, x_j = 0$ which is impossible.

Definition 5.2 Let (t_i) and (s_i) be given as above and $\alpha, p > 0$. A sequence $x = (x_j)$ is called to be strongly p −deferred Cesàro summable of order α to σ if

$$\lim_{i\to\infty} \frac{1}{s_i - {t_i}^\alpha} \sum_{t_i+1}^{s_i} \left|x_j - \sigma\right|^p = 0 \tag{5.2}$$

and this is denoted by $D_d^\alpha - lim\, x_j = \sigma$. The set of all strongly p −deferred Cesàro summable sequences of order α have been denoted by D_d^α.

Theorem 5.3 Let $0 < \alpha \leq 1$ and $x = (x_j)$, $y = (y_j)$ be sequences of complex numbers.

i) If $S_d^\alpha - \lim x_j = x$ and $c \in \mathbb{R}$ then $S_d^\alpha - \lim cx_j = cx$,

ii) If $S_d^\alpha - \lim x_j = x$ and $S_d^\alpha - \lim y_j = y$, then $S_d^\alpha - \lim(x_j + y_j) = x + y$,

iii) If $D_d^\alpha - \lim x_j = x$ and $c \in \mathbb{R}$ then $D_d^\alpha - \lim cx_j = cx$,

iv) If $D_d^\alpha - \lim x_j = x$ and $D_d^\alpha - \lim y_j = y$ then $D_d^\alpha - \lim(x_j + y_j) = x + y$,

Proof. Omitted

Theorem 5.4 Let $\alpha, \beta \in (0,1] (0 < \alpha \leq \beta \leq 1)$, then $S_d^\alpha \subseteq S_d^\beta$ and the inclusion is strict.

Proof. The inclusion part of the proof follows from the following inequality:

$$\frac{1}{(s_i - t_i)^\beta} \left|\{j : t_i < j \leq s_i, \left|x_j - \sigma\right| \geq \varepsilon\}\right|$$

$$\leq \frac{1}{(s_i - t_i)^\alpha} \left|\{j : t_i < j \leq s_i, \left|x_j - \sigma\right| \geq \varepsilon\}\right| .$$

To show the inclusion is strict, let us define a sequence by

$$x_j = \begin{cases} 1, & j = i^2 \\ 0, & j \neq i^2 \end{cases} .$$

Then $x \in S_d^{\beta}$ for $\frac{1}{2} < \beta \leq 1$, but $x \notin S_d^{\alpha}$ for $0 < \alpha \leq \frac{1}{2}$, where $s_i = 3i - 1$ and $t_i = 2i - 1$.

Theorem 5.5 If $\lim_i \frac{(s_i - t_i)^{\alpha}}{i} > 0$ then $S \subset S_d^{\alpha}$.

Proof . Let $S - \lim x_j = L$ and $\lim_i \frac{(s_i - t_i)^{\alpha}}{i} > 0$. For a given $\varepsilon > 0$, we have

$\{j \leq i: |x_j - \sigma| \geq \varepsilon\} \supseteq \{j: t_i < j \leq s_i, |x_j - \sigma| \geq \varepsilon\}$.

Therefore

$$\frac{1}{i}\left|\{j \leq i: |x_j - \sigma| \geq \varepsilon\}\right| \geq \frac{1}{i}\left|\{j: t_i < j \leq s_i, |x_j - \sigma| \geq \varepsilon\}\right|$$

$= \frac{(s_i - t_i)^{\alpha}}{i} \frac{1}{(s_i - t_i)^{\alpha}}\left|\{j: t_i < j \leq s_i, |x_j - \sigma| \geq \varepsilon\}\right|$.

Taking limit as $i \to \infty$ and using the fact that $\lim_i \frac{(s_i - t_i)^{\alpha}}{i} > 0$, we get $S_d^{\alpha} - \lim x_k = \sigma$.

Theorem 5.6 Let $\alpha, \beta \in (0,1] (0 < \alpha \leq \beta \leq 1)$, and $p > 0$, then $D_d^{\alpha} \subseteq D_d^{\beta}$ and the inclusion is strictness.

Proof. The inclusion section of the proof follows from the next inequality:

$\frac{1}{(s_i - t_i)^{\beta}} \sum_{t_i+1}^{s_i} |x_j - \sigma|^p \leq \frac{1}{(s_i - t_i)^{\alpha}} \sum_{t_i+1}^{s_i} |x_j - \sigma|^p$.

For the converse, let

$x_j = \begin{cases} 1, & j = i^2 \\ 0, & j \neq i^2 \end{cases}$.

Then

$$\frac{1}{(s_i - t_i)^{\beta}} \sum_{t_i+1}^{s_i} |x_j - \sigma|^p = \frac{1}{(s_i - t_i)^{\beta}} \sum_{t_i+1}^{s_i} |x_j - 0|^p$$

$$\leq \frac{\sqrt{s_i} - \sqrt{t_i} + 1}{(s_i - t_i)^{\beta}} \to 0.$$

Therefore we have $x \in D_d^{\beta}$ for $\frac{1}{2} < \beta \leq 1$, but

$$\frac{\sqrt{s_i} - \sqrt{t_i} - 1}{(s_i - t_i)^{\beta}} \leq \frac{1}{(s_i - t_i)^{\alpha}} \sum_{t_i+1}^{s_i} \left|x_j - 0\right|^p \to \infty$$

Then $x \notin D_d^{\alpha}$ for $0 < \alpha \leq \frac{1}{2}$.

Theorem 5.7 Let $0 < \alpha \leq 1$ and $0 < n < m < \infty$, then $D_d^{\alpha}(m) \subseteq D_d^{\alpha}(n)$.

Proof. Omitted.

Theorem 5.8 Let $\alpha, \beta \in (0,1] (0 < \alpha \leq \beta \leq 1)$, and $0 < p < \infty$, then $S_d^{\beta} \subset D_d^{\alpha}$.

Proof. For $\varepsilon > 0$, we can write

$$\sum_{t_i+1}^{s_i} \left|x_j - \sigma\right|^p = \sum_{\substack{t_i+1 \\ |x_j - \sigma| \geq \varepsilon}}^{s_i} \left|x_j - \sigma\right|^p + \sum_{\substack{t_i+1 \\ |x_j - \sigma| < \varepsilon}}^{s_i} \left|x_j - \sigma\right|^p$$

$$\geq \sum_{\substack{t_i+1 \\ |x_j - \sigma| \geq \varepsilon}}^{s_i} \left|x_j - \sigma\right|^p$$

$$\geq \left|\left\{t_i < k \leq s_i : \left|x_j - \sigma\right| \geq \varepsilon\right\}\right| . \varepsilon^p$$

and so

$$\frac{1}{(s_i - t_i)^{\alpha}} \sum_{t_i+1}^{s_i} \left|x_j - \sigma\right|^p \geq \frac{1}{(s_i - t_i)^{\alpha}} \left|\left\{t_i < k \leq s_i : \left|x_j - \sigma\right| \geq \varepsilon\right\}\right| . \varepsilon^p$$

$$\geq \frac{1}{(s_i - t_i)^{\beta}} \left|\left\{t_i < k \leq s_i : \left|x_j - \sigma\right| \geq \varepsilon\right\}\right| \varepsilon^r .$$

Even if $x = (x_j)$ is a bounded sequence, the converse of Theorem 5.8 doesn't hold. To indicate this we have to find a sequence that bounded and deferred statistically convergent of order α but doesn't need to be strongly p −deferred Cesàro summable of order β. To show this let $t_i = 0$ and $s_i = i$ for all $i \in \mathbb{N}$ and $x = (x_j)$ be defined as follows

$$x_j = \begin{cases} \dfrac{1}{\sqrt{j}}, & j \neq m^3, \\ 1, & j \neq m^3. \end{cases}$$

It can be shown that $x \in l_\infty \cap D_d^\alpha$ for $\alpha \in \left(\frac{1}{3}, 1\right]$. First of all, recall that the inequality $\sum_{j=1}^{i} \frac{1}{\sqrt{j}} > \sqrt{i}$ is satisfied for $i \geq 2$. Define $H_i = \{t_i < j \leq s_i : j \neq m^3, m = 1,2,3, \dots\}$ and take $p = 1$. Since

$$\sum_{t_i+1}^{s_i} |x_j|^p = \sum_{j=1}^{i} |x_j| = \sum_{j \in H_i}^{i} |x_k| + \sum_{j \notin H_i}^{i} |x_j|$$

$$= \sum_{j \in H_i}^{i} \frac{1}{\sqrt{j}} + \sum_{j \notin H_i}^{i} 1 > \sum_{j=1}^{i} \frac{1}{\sqrt{j}} > \sqrt{j}$$

we have

$$\frac{1}{(s_i - t_i)^\alpha} \sum_{t_i+1}^{s_i} |x_j|^p = \frac{1}{i^\alpha} \sum_{j=1}^{i} |x_j|$$

$> \frac{1}{i^\alpha}\sqrt{i} = \frac{1}{i^{\alpha - \frac{1}{2}}} \to \infty \; as \; n \to \infty$ for $\alpha \in \left(0, \frac{1}{2}\right)$.

So $x \notin D_d^\alpha$ for $\alpha \in \left(0, \frac{1}{2}\right)$ and therefor $x \in S_d^\alpha - D_d^\alpha$ for $\alpha \in \left(\frac{1}{3}, \frac{1}{2}\right)$.

The next results are obtained from Theorem 5.4, Theorem 5.5, Theorem 5.7 and Theorem 5.8.

Corollary 5.9 Let $\alpha \in (0,1]$, then;

i) $S_d^\alpha \subseteq S_d$ and the inclusion is strict,

ii) If $\lim_i \frac{s_i - t_i}{i} > 0$ then $S \subset S_d$,

iii) $Dw_r^\alpha[p,q] \subseteq Dw_r\ [p,q]$ and the inclusion is strict,

iv) $D_d^\alpha \subseteq D_d$

vi) $D_d^\alpha \subseteq S_d$

vii) $D_d \subseteq S_d$

Definition 5.10 Let $\alpha \in (0,1]$, $p > 0$ and $\{\mu(n)\}$ be an increasing sequence of positive integers with $\mu(0) = 0$. A sequence $x = (x_j)$ is called to be strongly $(p - C_\mu)$ −summable of order α to σ (or $(p - C_\mu^\alpha)$ −summable), if

$$\lim_{n\to\infty} \frac{1}{\mu(n)^\alpha} \sum_{1}^{\mu(n)} |x_j - \sigma| = 0,$$

this is denoted by $(p - C_\lambda^\alpha) - \lim x_k = \sigma$, where
$(\mu(n)^\alpha) = (\mu(1)^\alpha, \mu(2)^\alpha, \mu(3)^\alpha, \dots, \mu(n)^\alpha, \dots)$. The strongly $(p - C_\mu^\alpha)$ −summability reduces to the strongly $(p - C_\mu\ )$ −summability for $\alpha = 1$.

Theorem 5.11 Let $\in (0,1], p > 0$, $\{\mu(n)\}$ be defined as above and $s(n) = \mu(n), t(n) = \mu(n-1)$. If $D_d^\alpha - \lim x_k = \sigma$ and so$(-C_\lambda^\alpha) - \lim x_k = \sigma$.

Proof. Suppose that (x_k) is strongly p −deferred Cesàro summable of order α to L, and so we have

$$\frac{1}{\mu(n)^\alpha} \sum_{1}^{\mu(n)} |x_k - \sigma|^p = \sum_{i=0}^{n-1} \left(\frac{1}{\mu(n)^\alpha} \sum_{\mu(i)+1}^{\mu(i+1)} |x_k - \sigma|^p \right)$$

$$= \sum_{i=0}^{n-1} \left(\frac{(\mu(i+1) - \mu(i))^{\alpha}}{\mu(n)^{\alpha}(\mu(i+1) - \mu(i))^{\alpha}} \sum_{\mu(i)+1}^{\mu(i+1)} |x_k - \sigma|^p \right)$$

$$= \sum_{i=0}^{n-1} \left(\frac{(\mu(i+1) - \mu(i))^{\alpha}}{\mu(n)^{\alpha}} \sum_{\mu(i)+1}^{\mu(i+1)} \frac{1}{(\mu(i+1) - \mu(i))^{\alpha}} |x_k - \sigma|^p \right)$$

$$= \sum_{i=0}^{n-1} a_{ni} \, (D_d^{\alpha} x)_i$$

where

$$a_{ni} = \begin{cases} \dfrac{(\mu(i+1) - \mu(i))^{\alpha}}{\mu(n)^{\alpha}}, & i = 1,2,3, \dots, n-1, \\ 0, & otherwise. \end{cases}$$

Since, $A = (a_{ni})$ matrix is regular and $\lim_{i\to\infty} (D_d^{\alpha} x)_i = 0$ we get $\lim_{n\to\infty} \frac{1}{\mu(n)^{\alpha}} \sum_{1}^{\mu(n)} |x_k - \sigma|^p = 0$. Hence $x = (x_k)$ is strongly $p - C_{\mu}$ –summable of order α to σ.

CONCLUSION

The concepts of statistical convergence and strongly Cesaro summability of sequence of complex (or real) numbers have been studied by various mathematicians. In this work we introduce the concepts of deferred statistically convergent sequences of order α and strong p$-$deferred Cesaro summable sequences of order α and given and some relation between deferred statistically convergent sequence of order α and strong p–deferred Cesaro summable sequence of order α.

REFERENCES

[1] **Zygmund, A.** Trigonometric Series, 2nd edn. Cambridge University Press, Cambridge (1979)

[2] **Schoenberg, I. J.** The integrability of certain functions and related summability methods. Am. Math. Mon. 66, 361-375 (1959)

[3] **Fast, H.** Sur la convergence statistique. Colloq. Math. 2, 241-244 (1951)

[4] **Steinhaus, H.** Sur la convergence ordinaireet la convergence asymptotique. Colloq.Math. 2, 73-74 (1951)

[5] **Buck**, **R. C.** Generalized asymptotic density, Amer. J. Math., 75(1953), 335-346.

[6] **Šalàt, T.** On statistically convergent sequences of real numbers. Math. Slovaca 30(2), 139-150 (1980)

[7] **Fridy, J. A.** On statistical convergence. Analysis 5, 301-313 (1985)

[8] **Connor, J. S.** The statistical and strong p-Cesàro convergence of sequences. Analysis 8, 47-63 (1988).

[9] **Mursaleen, M.** λ-Statistical convergence. Math.Slovaca 50(1), 111-115 (2000).

[10] **Savas, E.** Strong almost convergence and almost λ-statistical convergence. Hokkaido Math. J. 29, 531-536 (2000)

[11] **Et, M. and Nuray, F**. Δ^m-statistical convergence. Indian J. Pure Appl. Math. 32 (2001), no. 6, 961--969.

[12] **Çolak, R.** Statistical convergence of order α. In: Modern Methods in Analysis and Its Applications, pp. 121-129. Anamaya Pub., New Delhi (2010)

[13] **Bhardwaj, V. K. and Gupta, S.** On some generalizations of statistical boundedness. *J. Inequal. Appl.* 2014**,** 2014:12, 11 pp

[14] **Bhardwaj, V. K.** ; **Gupta, S.** ; **Mohiuddine, S. A. and Kılıçman, A.** On lacunary statistical boundedness, J. Inequal. Appl**.** 2014**,** 2014:311, 11 pp.

[15] **Çolak, R.** On λ-statistical convergence. In: Conference on Summability and Applications, ˙Istanbul, Turkey, May 12-13 (2011).

[16] **Çolak, R. and Bektas, Ç. A.** λ-Statistical convergence of order α. Acta Math. Sin. Engl. Ser. 31(3), 953-959 (2011).

[17] **Connor**, **J. S.** On strong matrix summability with respect to a modulus and statistical convergence, Canad. Math. Bull., 32(1989), 194-198.

[18] **Fridy, J. A.** Statistical limit points. Proc. Am. Math. Soc. 118, 1187-1192 (1993).

[19] **Fridy, J. A. and Orhan, C.** Lacunary statistical convergence. Pac. J. Math. 160, 43-51 (1993).

[20] **Fridy, J. A. and Orhan, C.** Statistical limit superior and limit inferior. Proc. Am. Math. Soc. 125(12), 3625-3631 (1997).

[21] **Et, M. ; Çinar, M. and Karakas, M.** On λ-statistical convergence of order α of sequences of function. J. Inequal. Appl. 2013, Article ID 204 (2013)

[22] **Et, M.** Generalized Cesàrodifference sequence spaces of non-absolute type involving lacunary sequences, Appl. Math. Comput.219 (2013), no. 17, 9372–9376.

[23] **Et, M. ; Altınok, H. and Çolak, R.** On λ statistical convergence of difference sequences of fuzzy numbers, Inform. Sci. 176(15) (2006), 2268–2278.

[24] **Et, M. and Şengül, H.** On (Δ^m, I)-lacunary statistical convergence of order α. J. Math. Anal. **7** (2016), no. 5, 78-84.

[25] **Gadjiev, A. D. and Orhan,** C, 2002, Some approximation theorems via statistical convergence, Rocky Mountain J. Math, 32(1), pp. 129–138.

[26] **Isik, M.** Strongly almost $(w, \lambda, q)-$summable sequences, *Math.* Slovaca 61(5) (2011), 779–788.

[27] **Isik, M**. Generalized vector-valued sequence spaces defined by modulus functions, J. Inequal. Appl. 2010, Art.ID 457892, 7 pp.

[28] **Mursaleen, M.** and **Mohiuddine, S. A.** Korovkin-type approximation theorem for functions of two variables via statistical summability (C,1,1). Analytic number theory, approximation theory, and special functions, *549-560*, Springer, New York, *2014.*

[29] **Mursaleen, M. and Mohiuddine, S. A**. Korovkin type approximation theorem for almost and statistical convergence. *Nonlinear analysis*, 487-494, Springer Optim. *Appl., 68,* Springer, New York, 2012.

[30] **Mursaleen, M.** ; **Ahmad, R.** Korovkin type approximation theorem through statistical lacunary summability. Iran. J. Sci. Technol. Trans. A Sci. **37** (2013), no. 2, 99-102.

[31] **Savas, E.** On statistically convergent sequences of fuzzy numbers. Inf. Sci. 137(1-4), 277-282 (2001)

[32] **Savas, E.** On asymptotically λ-statistical equivalent sequences of fuzzy numbers. New Math. Nat. Comput. 3(3), 301-306 (2007)

[33] **Savas, E**. On λ-statistically convergent double sequences of fuzzy numbers. J. Inequal. Appl. 2008, Article ID 147827 (2008).

[34] **Rath, D. and Tripathy, B. C.** On statistically convergent and statistically Cauchy sequences. Indian J. Pure Appl. Math. 25(4), 381-386 (1994).

[35] **Temizsu, F.** ; **Et, M. and Çınar, Muhammed**. Δ^{m}-deferred statistical convergence of order α. Filomat 30 (2016), no. 3, 667-673.

[36] **Agnew R. P.**, On deferred Cesaro Mean, Comm. Ann. Math., 33(1932), 413-421.

[37] **Yilmaztürk, M. and Küçükaslan, M.** On strongly deferred Cesàro summability and deferred statistical convergence of the sequences, Bitlis Eren Univ. J. Sci. and Technol. 3 (2011), 22-25.

[38] **Küçükaslan, M. and Yılmaztürk, M.** On deferred statistical convergence of sequences. *Kyungpook Math. J.* 56 (2016), no. 2, 357-366.

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