REPUBLIC OF TURKEY
FIRAT UNIVERSITY
THE GRADUATE SCHOOL OF NATURAL AND APPLIED SCIENCES
DEPARTMENT OF MATHEMATICS

# STABILITY OF THE ORDINARY DIFFERENTIAL EQUATIONS AND SOME APPLICATIONS

**Master Thesis**

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**JUNE – 2017**

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# CONTENTS

## SUMMARY

### Stability of the ordinary differential equations and some applications

In this thesis consists of five chapters. In the first an introduction then define differential equation, such as types of differential equation, then explained the system of ordinary differential equations, described on non-linear differential equations and Lyapunov's theory of stability. Then, application are given as well. Also about application in biology and SEIR model, the basics of SEIR model explained. Some fundamental conclusions about thesis are given.



**Keywords:** Ordinary differential equations, Lyapunov's theory of stability, SEIR model.

## ÖZET


**Adi diferensiyel denklemlerin kararlılığı ve bazı uygulamaları**

Bu tez beş bölümden oluşmaktadır. Birinci bölümde giriş az şekilde tarihçe ye almaktadır. İkinci bölümde diferensiyel denklem tanımı yapılmış, bazı denklem örnekleri verilmiştir. Bu bölümde birinci basamaktan sistemler ve örnekler bulunmaktadır. Üçüncü bölümde nonlineer diferensiyel denklemlerden bahsedilmiştir. Dördüncü bölümde Lyapunov'un kararlılık teorisi bulunmaktadır. Beşinci ve son bölüm kimyasal kinetikten ve SEIR modelinden bahsedilmiş SEIR modeli anlatılmış ve örnek verilmiştir.



**Anahtar Kelimeler:** Adi diferensiyel denklemler, Lyapunov teorisi, SEIR modeli.

## 1. INTRODUCTION

Differential equations have received a considerable amount of interest due to its broad applications. Ordinary differential equations play an important role in many branches of applied and pure mathematics and their applications in engineering, applied mechanics, quantum physics, analytical chemistry, astronomy and biology. From last decade, researcher pay attentions towards analytical and numerical solutions of nonlinear ordinary differential equations. Therefore, it becomes increasingly important to be familiar with all traditional and recently developed methods for solving linear and nonlinear ordinary differential equations**.** (See [5], [6], and [7]). Lyapunov's theory second (or direct) method provides tools for studying (asymptotic) stability properties of an equilibrium point of a dynamical system (or systems of differential equations). (See [9], [10]).

Mathematical biology seems to have grown exponentially starting in the applied mathematics of the 21$^{th}$ century. A huge variety of models have been formulated, mathematically analyzed and applied to infectious diseases. The SEIR model with nonlinear incidence rates in epidemiology is studied, an SEIR model with limited resources for treatment. Method for analyzing a general compartmental. (See [15], [19], and [22]).

## 2. DIFFERENTIAL EQUATION AND SYSTEM OF ORDINARY DİFFERENTIAL EQUATION

The first definition that we should cover should be that of differential equation ([1],[2], [3]).

### 2.1 Differential equation

An equation involving derivatives of one or more dependent variables with respect to one or more independent variables is called a differential equation.

**Definition 2.1.1. (Ordinary differential equation)** A differential equation involving ordinary derivatives of one or more dependent variables with respect to a single independent variable is called an ordinary differential equation.

**Example 2.1.1.** $\frac{d^2y}{dx^2} + xy(\frac{dy}{dx})^2 = 0$ , $\frac{d^4x}{dt^4} + 5\frac{d^2y}{dt^2} + 3x = \sin t.$

**Definition 2.1.2. (Partial differential equation)** A differential equation involving partial derivatives of one or more dependent variables with respect to more than one independent variable is called a partial differential equation.

**Example 2.1.2.** $\frac{\partial v}{\partial s} + \frac{\partial v}{\partial t} = v$ , $\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2} = 0.$

**Definition 2.1.3. (Linear ordinary differential equation)** A linear ordinary differential equation of order $n$, in the dependent variable y and the independent variable $x$, is an equation that is in, or can be expressed in, the form

$$a_0(x)\frac{d^ny}{dx^n} + a_1(x)\frac{d^{n-1}}{dx^{n-1}} + \cdots + a_{n-1}(x)\frac{dy}{dx} + a_n(x)y = b(x),$$

where $a_0$ is not identically zero.

**Example 2.1.3.** The following ordinary differential equations are both linear.In each case $y$ is the dependent variable. Observe that $y$ and its various derivatives occur to the first degree only and that no products of $y$ and of its derivatives are present.

$$\frac{d^2y}{dx^2} + 5\frac{dy}{dx} + 6y = 0 \,,$$

$$\frac{d^4y}{dx^4} + x^2\frac{d^3y}{dx^3} + x^3\frac{dy}{dx} = xe^x.$$

**Definition 2.1.4. Nonlinear ordinary differential equation**

A nonlinear ordinary differential equation is an ordinary differential equation that is not linear.

**Example 2.1.4.** The following ordinary differential equation are all nonlinear:

$$\frac{d^2y}{dx^2} + 5\frac{dy}{dx} + 6y^2 = 0 \,,$$

$$\frac{d^2y}{dx^2} + 5\left(\frac{dy}{dx}\right)^3 + 6y = 0 \,,$$

$$\frac{d^2y}{dx^2} + 5y\frac{dy}{dx} + 6y = 0 \,.$$

**2.2. System of ordinary differential equation**

In the solution of many problems it is required to find the function $y_1 = y_1(x)$, $y_2 = y_2(x)$, …., $y_n = y_n(x)$, which satisfy a system of differential equations containing the argument $x$, the unknown function $y_1, y_2, \ldots., y_n$ and their derivatives [9].

Consider the following system of first-order equations:

$$\left.\begin{array}{l} \frac{dy_1}{dx} = f_1(x, y_1, y_2, \ldots., y_n) \\ \frac{dy_2}{dx} = f_2(x, y_1, y_2, \ldots., y_n) \\ \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \\ \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \\ \frac{dy_n}{dx} = f_n(x, y_1, y_2, \ldots, y_n) \end{array}\right\} \qquad (2.2.1)$$

where $y_1, y_2, \ \ldots, \ y_n$ unknown function and $x$ is the argument.

A system of this kind, where the left sides of the equations contain first-order derivatives, while the right sides do not contain derivatives, is called *normal*.

To integrate the system means to determine the functions $y_1, y_2, \dots, y_n$, which satisfy the system of equation (2.2.1) and the given initial conditions:

$$(y_1)_{x=x_0} = y_{10}\,,\quad (y_2)_{x=x_0} = y_{20}\,,\dots\dots,\quad (y_n)_{x=x_0} = y_{n0} \tag{2.2.2}$$

Integration of a system like (2.2.1) is performed as follows. Differentiate the first equation of (2.2.1) with respect to $x$:

$$\frac{d^2y_1}{dx^2} = \frac{\partial f_1}{\partial x} + \frac{\partial f_1}{\partial y_1}\frac{\partial y_1}{\partial x} + \dots + \frac{\partial f_1}{\partial y_n}\frac{\partial y_n}{\partial x}$$

Replacing the derivatives $\frac{dy_1}{dx}$ , $\frac{dy_2}{dx}$ ,…., $\frac{dy_n}{dx}$ with their expressions $f_1$ , $f_2$ , …., $f_n$ from equations (2.2.1), we get the equation

$$\frac{d^2y_1}{dx^2} = F_2\,(x, y_1, \dots, y_n)$$

Differentiating this equation and then doing as before, we obtain

$$\frac{d^3y_1}{dx^3} = F_3\,(x,\, y_1, \dots,\, y_n).$$

Continuing in the same fashion, we finally get the equation

$$\frac{d^ny_1}{dx^n} = F_n\,(x,\, y_1\,, \dots,\, y_n).$$

We thus get the following system:

$$\left.\begin{aligned} &\frac{dy_1}{dx} = f_1(x, y_1, y_2, \dots, y_n) \\ &\frac{d^2y_2}{dx^2} = F_2(x, y_1, y_2, \dots, y_n) \\ &\dots\dots\dots\dots\dots\dots\dots\dots\dots \\ &\dots\dots\dots\dots\dots\dots\dots\dots\dots \\ &\frac{d^ny_1}{dx^n} = F_n(x, y_1, y_2, \dots, y_n). \end{aligned}\right\} \tag{2.2.3}$$

From the first $n-1$ equations we determine (if this is possible) $y_1, y_2, \ldots, y_n$ and express them in terms of $x, y_1$ and the derivatives $\frac{dy_1}{dx}, \frac{d^2y_1}{dx^2}, \ldots, \frac{d^{n-1}y_1}{dx^{n-1}}$ :

$$\left.\begin{array}{l} y_2 = \varphi_2\left(x, y_1, y_1', y_1^{(n-1)}\right) \\ y_3 = \varphi_3\left(x, y_1, y_1', y_1^{(n-1)}\right) \\ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \\ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \\ y_n = \varphi_n\left(x, y_1, y_1', y_1^{(n-1)}\right). \end{array}\right\} \tag{2.2.4}$$

Putting these expressions into the last of the equations (2.2.3), we get an *n*th-order equation for determining $y_1$ :

$$\frac{d^n y_1}{dx^n} = \Phi\left(x, y_1, y_1', \ldots, y_1^{(n-1)}\right). \tag{2.2.5}$$

Solving this equation, we find $y_1$ :

$$y_1 = \psi_1(x, C_1, C_2, \ldots, C_n)\ . \tag{2.2.6}$$

Differentiating the latter expression $n-1$ times, we find the derivatives

$$\frac{dy_1}{dx}, \frac{d^2y_1}{dx^2}, \ldots, \frac{d^{n-1}y_1}{dx^{n-1}} \text{ as functions of } x, C_1, C_2, \ldots, C_n\ .$$

Substituting these functions into equations (2.2.4), we determine $y_2, y_3, \ldots, y_n$ :

$$\left.\begin{array}{l} y_2 = \psi_2(x, C_1, C_2, \ldots, C_n) \\ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \\ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \\ y_n = \psi_n(x, C_1, C_2, \ldots, C_n) \end{array}\right\} \tag{2.2.7}$$

For this solution to satisfy the given initial conditions (2.2.2), it remains for us to find [from equations (2.2.6) and (2.2.7)] the appropriate values of the constants $C_1, C_2, \ldots, C_n$ (like we did in the case of a single differential equation).

**Note 2.2.1.** If the system (2.2.1) is linear in the unknown function, then equation (2.2.5) is also linear.

**Example 2.2.1.** Integrate the system

$$\left.\begin{aligned}\frac{dy}{dx} &= y + z + x \\ \frac{dz}{dx} &= -4y - 3z + 2x\end{aligned}\right\} \qquad (a)$$

With the initial conditions

$$(y)_{x=0} = 1\,, \quad (z)_{x=0} = 0. \qquad (b)$$

**Solution.**

**i.** Differentiating the first equation with respect to $x$, we have

$$\frac{d^2y}{dx^2} = \frac{dy}{dx} + \frac{dz}{dx} + 1$$

Putting the expressions $\frac{dy}{dx}$ and $\frac{dz}{dx}$ from equations ($a$) into this equation, we get

$$\frac{d^2y}{dx^2} = (y + z + x) + (-4y - 3z + 2x) + 1$$

or

$$\frac{d^2y}{dx^2} = -3y - 2z + 3x + 1 \qquad (c)$$

**ii.** From the first equation of system (a) we find

$$z = \frac{dy}{dx} - y - x \qquad (d)$$

And put it into the equation just obtained; we get

$$\frac{d^2y}{dx^2} = -3y - 2\left(\frac{dy}{dx} - y - x\right) + 3x + 1$$

or

$$\frac{d^2y}{dx^2} + 2\frac{dy}{dx} + y = 5x + 1 \qquad (e)$$

The general solution of this equation is

$$y = (C_1 + C_2 x)e^{-x} + 5x - 9 \qquad (f)$$

And from (*d*) we have

$$z = (C_2 - 2C_1 - 2C_2 x)e^{-x} - 6x + 14 \qquad (g)$$

Choosing the constants $C_1$ and $C_2$ so that the initial conditions (*b*) are satisfied,

$$(y)_{x=0} = 1\,,\ (z)_{x=0} = 0.$$

We get, from equation (*f*) and (*g*),

$$1 = C_1 - 9\,,\ \ 0 = C_2 - 2C_1 + 14,$$

whence $C_1 = 10$ and $C_2 = 6$ .

Thus, the solution that satisfies the given initial conditions (*b*) has the form

$$y = (10 + 6x)e^{-x} + 5x - 9\,,\quad z = (-14 - 12x)e^{-x} - 6x + 14.$$

**Example 2.2.2.** Integrate the system $\frac{dx}{dt} = y + z$ ,

$$\frac{dy}{dt} = x + z\,,$$

$$\frac{dz}{dt} = x + y\,.$$

**Solution.** Differentiating the first equation with respect to $t$, we find

$$\frac{d^2x}{dt^2} = \frac{dy}{dt} + \frac{dz}{dt} = (x + z) + (x + y)$$

$$\frac{d^2x}{dt^2} = 2x + y + z$$

Eliminating the variables $y$ and $z$ from the equations

$$\frac{dx}{dt} = y + z \ , \quad \frac{d^2x}{dt^2} = 2x + y + z$$

We get a second-order equation in $x$:

$$\frac{d^2x}{dt^2} - \frac{dx}{dt} - 2x = 0$$

Integrating this equation, we obtain its general solution:

$$x = C_1e^{-t} + 2C_2e^{2t} \qquad (a)$$

Whence we find

$$\frac{dx}{dt} = -C_1e^{-t} + 2C_2e^{2t} \quad \text{and} \quad y = \frac{dx}{dt} - z = -C_1e^{-t} + 2C_2e^{2t} - z \qquad (b)$$

Putting into the third of the given equation the expressions that have been found for $x$ and $y$, we get an equation for determining $z$:

$$\frac{dz}{dt} + z = 3C_2e^{2t}$$

Integrating this equation, we find

$$z = C_3e^{-t} + C_2e^{2t} \qquad (c)$$

But then, from equation ($b$), we get $$y = -(C_1 + C_3)e^{-t} + C_2e^{2t} \qquad (d)$$

Equation ($a$), ($c$), and ($d$) give the general solution of the given system.

The differential equations of a system may contain higher-order derivatives. This then yields a system of differential equations of higher order.

For instance, the problem of the motion of a material point under the action of a force $F$ reduces to a system of three second-order differential equations, Let $F_x, F_y, F_z$ be the projections of the force $F$ on the coordinate axes. The position of the point at any instant of time $t$ is determined by its coordinates $x, y$, and $z$. Hence, $x, y, z$ are functions of $t$, the projections of the velocity vector of the point on the axes will be $\frac{dx}{dt}$, $\frac{dy}{dt}$, $\frac{dz}{dt}$.

Suppose that the force $F$ and, hence, its projections $F_x, F_y$, $F_z$ depend on the time $t$, the coordinates $x, y, z$ of the point, and on the velocity of motion of the point, that is on $\frac{dx}{dt}, \frac{dy}{dt}, \frac{dz}{dt}$.

In this problem the following three functions are the sought-for functions:

$$x = x(t),\ y = y(t),\ z = z(t).$$

These functions are determined from equations of dynamics (Newton's law):

$$\left.\begin{array}{l} m\frac{d^2x}{dt^2} = F_x\ (t, x, y, z, \frac{dx}{dt}, \frac{dy}{dt}, \frac{dz}{dt}) \\ m\frac{d^2y}{dt^2} = F_y\ (t, x, y, z, \frac{dx}{dt}, \frac{dy}{dt}, \frac{dz}{dt}) \\ m\frac{d^2z}{dt^2} = F_z\ (t, x, y, z, \frac{dx}{dt}, \frac{dy}{dt}, \frac{dz}{dt}) \end{array}\right\} \tag{2.2.8}$$

We thus have a system of three of second-order differential equations. In the case of plane motion, that is, motion in which the trajectory is a plane curve (lying, for example, in the $xy-$ plane), we get a system of two equations for determining the functions $x(t)$ and $y(t)$ :

$$m\frac{d^2x}{dt^2} = F_x\ (t, x, y, \frac{dx}{dt}, \frac{dy}{dt}) \tag{2.2.9}$$

$$m\frac{d^2y}{dt^2} = F_y\ (t, x, y, \frac{dx}{dt}, \frac{dy}{dt}) \tag{2.2.10}$$

It is possible to solve a system of differential equations of higher order by reducing it to a system of first-order equations. Using equations (2.2.9) and (2.2.10) as examples, we shall show how this is done. We introduce the notation

$$\frac{dx}{dt} = u\ ,\quad \frac{dy}{dt} = v,$$

Then

$$\frac{d^2x}{dt^2} = \frac{du}{dt} \ , \quad \frac{d^2y}{dt^2} = \frac{dv}{dt} \, .$$

The system of two second-order equations (2.2.9) and (2.2.10) in two unknown functions $x(t)$ and $y(t)$ is replaced by a system of four first-order equations in four unknown functions $x,\ y,\ u,$ and $v$ :

$$\frac{dx}{dt} = u,$$

$$\frac{dy}{dt} = v,$$

$$m\frac{du}{dt} = F_x\ (t, x, y, u, v),$$

$$m\frac{dv}{dt} = F_y\ (t, x, y, u, v).$$

We remark in conclusion that the general method that we have considered of solving the system may, in certain specific cases, be replaced by some artificial technique that gets the result faster.

**Example 2.2.3.** Find the general solution of the following system of differential equations:

$$\frac{d^2y}{dx^2} = z,$$

$$\frac{d^2z}{dx^2} = y.$$

**Solution.** Differentiate both sides of the first equation twice with respect to $x$:

$$\frac{d^4y}{dx^4} = \frac{d^2z}{dx^2}$$

But $\frac{d^2z}{dx^2} = y$, and so we get a fourth-order equation

$$\frac{d^4y}{dx^4} = y$$

Integrating this equation, we obtain its general solution

$$y = C_1 e^{x} + C_2 e^{-x} + C_3 \cos x + C_4 \sin x$$

Finding $\frac{d^2y}{dx^2}$ from this equation and putting it into the first equation, we find $z$:

$$z = C_1 e^{x} + C_2 e^{-x} - C_3 \cos x - C_4 \sin x.$$

## 3. NON-LINEAR DIFFERENTIAL EQUATIONS

The mathematical formulation of numerous physical problems results in differential equations which are actually nonlinear. In many cases it is possible to replace such a nonlinear equation by a related linear equation which approximates the actual nonlinear equation closely enough to give useful results. However, such a "linearization" is not always feasible; and when is not, the original nonlinear equation itself must be considered. While the general theory and methods of linear equations highly developed, very little of a general character is known about nonlinear equations. The study of nonlinear equation is generally confined to a variety of rather special cases, a done must resort to a various methods of approximation. In this study we shall give a brief and useful introductions to certain of these methods. ([4], [5], [6], [7], [8]).

### 3.1 Basic concepts and definitions

In this section we shall be concerned with second-order nonlinear differential equations of the form $\frac{d^2x}{dt^2} = F(x, \frac{dx}{dt}).$
(3.1.1)

As a specific example of such an equation we list the important van der Pol equation

$$\frac{d^2x}{dt^2} + \mu(x^2 - 1)\frac{dx}{dt} + x = 0 \qquad (3.1.2)$$

where $\mu$ is a positive constant.

We shall consider this equation at a later stage of our study. For the time being, we merely note that (3.1.2) may be put in the form (3.1.1), where

$$F\left(x, \frac{dx}{dt}\right) = -\mu(x^2 - 1)\frac{dx}{dt} - x\,.$$

Let us suppose that the differential equation (3.1.1). Describes a certain dynamical system having one degree of freedom. The state of this system at time $t$ is determined by the values of $x$ (position), and $\frac{dx}{dt}$ is called a phase plane. If we let $y = \frac{dx}{dt}$, we can replace the second-order equation (3.1.1) by the equivalent system

$$\frac{dx}{dt} = y \,,$$
$$\frac{dy}{dt} = F(x, y) \,. \tag{3.1.3}$$

We can determine information about equation (3.1.1) from a study of the system (3.1.3). In particular we shall be interested in the configurations formed by the curves which the solutions of (3.1.3) define. We shall regard $t$ as a parameter so that these curves will appear in the $xy$ plane. Since $y = \frac{dx}{dt}$, this $xy$ plane is simply the $x$, $\frac{dx}{dt}$ phase plane mentioned in the preceding paragraph.

More generally, we shall consider system of the form

$$\frac{dx}{dt} = P(x, y) \,,$$
$$\frac{dy}{dt} = Q(x, y) \,, \tag{3.1.4}$$

where $P$ and $Q$ have continuous first partial derivatives for all $(x, y)$. Such a system, in which the independent variable $t$ appears only in the differentials $dt$ of the left members and not explicitly in the functions $P$ and $Q$ on the right, is called an autonomous system. We shall now proceed to study the configurations formed in the $xy$ phase plane by the curves which are defined by the solutions of (3.1.4). Any number $t_0$ and any pair $(x_0, y_0)$ of real numbers, there exists a unique solution

$$x = f(t) \,,$$
$$y = g(t) \,, \tag{3.1.5}$$

of the system (3.1.4) such that

$$f(t_0) = x_0 \,,$$

$$g(t_0) = y_0 \,.$$

If $f$ and $g$ are not both constant functions, then (3.1.5) defines a curve in the $xy$ plane which we shall call a *path* (*or orbit or trajectory*) of the system (3.1.4).

If the ordered pair of functions defined by (3.1.5) is a solution of (3.1.4) and $t_1$ is any real number, then it is easy to see that the ordered pair of functions defined by

$$x = f(t - t_1) \,, \qquad\qquad (3.1.6)$$
$$y = g(t - t_1) \,,$$

is also a solution of (3.1.4). Assuming that $f$ and $g$ in (3.1.5) are not both constant functions and that $t_1 \neq 0$, the solutions defined by (3.1.5) and (3.1.6) are two *different solutions* of (3.1.4). However, these two *different solutions* are simply different parameterizations of the *same path*. We thus observe that the terms *solution* and *path* are not synonymous. On the one hand, a *solution* of (3.1.4) is an ordered *pair of functions* $(f, g)$ such that $x = f(t)$ , $y = g(t)$ simultaneously satisfy the two equations of the system (3.1.4) identically ; on the other hand, a *path* of (3.1.4) is a *curve* in the $xy$ phase plane, which may be defined parametrically by more than one solution of (3.1.4).

Through any point of the $xy$ phase plane there passes at most one path of (3.1.4).

Let $C$ be a path of (3.1.4) and consider the totality of different solutions of (3.1.4) which define this path $C$ parametrically. For each of these defining solutions, $C$ is traced out in the same direction as the parameter $t$ increases. Thus with each path $C$. There is associated a definite direction, the direction of increase of the parameter $t$ in the various possible parametric representations of $C$ by the corresponding solutions of the system. In our figures we shall use arrows to indicate this direction associated with a path.

Eliminating $t$ between the two equations of the system (3.1.4), we obtain the equation

$$\frac{dy}{dx} = \frac{Q(x,y)}{P(x,y)} \ . \tag{3.1.7}$$

This equation gives the slope of the tangent to the path of (3.1.4) passing through the point$(x, y)$, provided the functions $P$ and $Q$ are not both zero at this point. The one-parameter family of solutions of (3.1.7) thus provides the one-parameter family of paths of (3.1.4). However, the description (3.1.7) does not indicate the directions associated with these paths.

At a point $(x_0, y_0)$ at which both $P$ and $Q$ are zero, the slope of the tangent to the path, as defined by (3.1.7), is indeterminate. Such points are singled out in the following definition.

**Definition 3.1.1.** Given the *autonomous* system

$$\frac{dx}{dt} = p(x,y) \ , \qquad \frac{dy}{dt} = Q(x,y) \ , \tag{3.1.4}$$

a point $(x_0, y_0)$ at which both

$$P(x_0, y_0) = 0 \ \text{ and } \ Q(x_0, y_0) = 0$$

is called a critical point of (3.1.4).

**Example 3.1.1.** Consider the linear autonomous system

$$\frac{dx}{dt} = y \ , \tag{3.1.8}$$

$$\frac{dy}{dt} = -x \ .$$

Differentiating to the system (3.1.8) with respect to $t$, we have

$$\frac{d^2x}{dt^2} = \frac{dy}{dt}$$

$$\frac{d^2y}{dt^2} = -\frac{dx}{dt}$$

$$\frac{d^2x}{dt^2} = -x \qquad \Longrightarrow \qquad \frac{d^2x}{dt^2} + x = 0$$

$$\lambda^2 + 1 = 0 \ \Longrightarrow \ \lambda_1 = i \ , \ \lambda_2 = -i$$

The general solution of $x(t)$ is $x = c_1 \sin t - c_2 \cos t$. By the same way we can find $y(t)$, the general solution of this system may be written:

$$x = c_1 \sin t - c_2 \cos t \ ,$$

$$y = c_1 \cos t + c_2 \ \sin t \ ,$$

Where $c_1$ and $c_2$ are arbitrary constants. The solution satisfying the conditions $x(0) = 0$, $y(0) = 1$ is readily found to be

$$x = \sin t \ , \qquad y = \cos t. \tag{3.1.9}$$

This solution defines a path $C_1$ in the $xy$ plane. The solution satisfying the conditions $x(0) = -1$, $y(0) = 0$ Is

$$x = \sin(t - \frac{\pi}{2}) \ , \qquad y = \cos(t - \frac{\pi}{2}) \ . \tag{3.1.10}$$

The solution (3.1.10) is different from the solution (3.1.9), but (3.1.10) also defines the same path $C_1$. That is, the ordered pairs of functions defined by (3.1.9) and (3.1.10) are two different solutions of (3.1.8) which are different parameterizations of the same path $C_1$. Eliminating $t$ from either (3.1.9) or (3.1.10) we obtain the equation $x^2 + y^2 = 1$ of the path $C_1$ in the $xy$ phase plane. Thus the path $C_1$ is the circle with center at (0,0) and radius 1. From either (3.1.9) or (3.1.10) we see that the direction associated with $C_1$ is the clockwise direction.

Eliminating $t$ between the equations of the system (3.1.8) we obtain the differential equation

$$\frac{dy}{dx} = -\frac{x}{y} \tag{3.1.11}$$

Which gives the slope of the tangent to the path of (3.1.8) passing through the point $(x, y)$, provided $(x, y) \neq (0,0)$.

The one-parameter family of solutions

$$x^2 + y^2 = c^2.$$

Of equation (3.1.11) gives the one-parameter family of paths in the $xy$ phase plane. Several of these are shown in (Figure 3.1). The path $C_1$ referred to above is, of course, That for which $c = 1$.

Looking back at the system (3.1.8), we see that $P(x, y) = y$ and $Q(x, y) = -x$. Therefore the only critical point of the system is the origin (0,0). Given any real number $t_0$,
The solution $x = f(t)$, $y = g(t)$ such that $f(t_0) = g(t_0) = 0$ is simply $x = 0$, $y = 0$, for all $t$.

We can also interpret the autonomous system (3.1.4) as defining a *velocity vector filed* $V$, where

$$V(x, y) = [P(x, y), Q(x, y)].$$

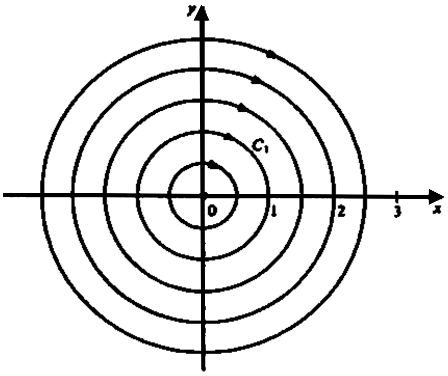


Figure 3.1

The $x$ component of this velocity vector at a point $(x, y)$ is given by $P(x, y)$, and $y$ component there is given by $Q(x, y)$. This vector is the velocity vector of representative point $R$ describing s path of (3.1.4) defined parametrically by a solution $x = f(t)$, $y = g(t)$. At a critical point both components of this velocity vector are zero, and hence at a critical point both point $R$ is at rest.

In particular, let us consider the special case (3.1.3), which arises from a dynamical system described by the differential equation (3.1.1). At a critical point of (3.1.3) both $dx/dt$ and $dy/dt$ are zero. Since

$$\frac{dy}{dt} = \frac{d^2x}{dt^2},$$

we thus see that at such a point the velocity and acceleration of the dynamical system described by (3.1.1) are both zero. Thus the critical points of (3.1.3) are equilibrium points of the dynamical system described by (3.1.1).

We now introduce certain basic concepts dealing with critical points and paths.

**Definition 3.1.2. (Isolated critical point)** A critical point $(x_0, y_0)$ of system (3.1.4) is called *isolated* if there exists a circle

$$(x - x_0)^2 + (y - y_0)^2 = r^2$$

About the point $(x_0, y_0)$ such that $(x_0, y_0)$ is the only critical point of (3.1.4) within this circle.

In what follows we shall assume that every critical point is isolated.

**Definition 3.1.3.** Let $C$ be a path of the system (3.1.4), and let $x = f(t)$ , $y = g(t)$ be a solution of (3.1.4) which resents $C$ parametrically. Let (0,0) be a critical point of (3.1.4). We shall say that the path $C$ approaches the critical point (0,0) as $t \to +\infty$ if

$$\lim_{t\to+\infty} f(t) = 0 , \quad \lim_{t\to+\infty} g(t) = 0 . \tag{3.1.12}$$

Thus when we say that a path $C$ defined parametrically by $x = f(t)$ , $y = g(t)$ approaches the critical point $(0,0)$ as $t \to +\infty$ , we understand the following: a point $R$ tracing out $C$ according to the equations $x = f(t), y = g(t)$ will approach the point $(0,0)$ as t$\to +\infty$ . This approach of a path $C$ to the critical point $(0,0)$ is independent of the solution actually used to represent $C$. That is, if $C$ approaches $(0,0)$as $t \to + + \infty$, then (3.1.12) is true for all solutions $x = f(t)$ , $y = g(t)$ representing $C$.

In like manner, a path $C_1$ approaches the critical point (0,0) $as\ t \to -\infty$ if

$$\lim_{t\to-\infty} f_1(t) = 0 , \quad \lim_{t\to-\infty} g_1(t) = 0$$

where $x = f_1(t)$ , $y = g_1(t)$ is a solution the path $C_1$.

**Definition 3.1.4.** Let $C$ be a path of the system (3.1.4) which approaches the critical point $(0,0)$ of (3.1.4) as $t \to +\infty$, and let $x = f(x)$ , $y = g(t)$ be a solution of (3.1.4) which represents $C$ parametrically. We say that $C$ enters the critical point $(0,0)$ as $t \to +\infty$, if

$$\lim_{t\to+\infty} \frac{g(t)}{f(t)} \tag{3.1.13}$$

Exists or if the quotient in (3.1.13) becomes either positively or negatively infinite as $t \to +\infty$.

We observe that the quotient $g(t)/f(t)$ in (3.1.13) represents the slope of the line joining critical point $(0, 0)$ and a point $R$ with coordinates $[f(t), g(t)]$ on $C$. Thus when we say that a path $C$ enters the critical point $(0, 0)$ as $t \to \infty$ we mean that the line joining $(0, 0)$ and a point $R$ tracing out $C$ approaches a definite "limiting" direction as $t \to +\infty$.

### 3.2 Types of Critical Points

We shall now discuss certain types of critical points which we shall encounter. We shall first give a geometric description of each types of critical point, referring to an appropriate figure as we do so; we shall then follow each such description with a more precise definition [7].

**Definition 3.2.1.** **(Center point)** The isolated critical point (0,0) of system (3.1.4) is called a *center* (Figure3.2), if there exists a neighborhood of (0,0) which contains a countable infinite number of closed paths $P_n (n = 1, 2, 3, \dots)$, each of which contains $(0, 0)$ in its interior, and which are such that the diameters of the paths approach 0 as $n \to \infty$ [but $(0, 0)$ is not approached by any path either as $t \to +\infty$ or as $t \to -\infty$ ].

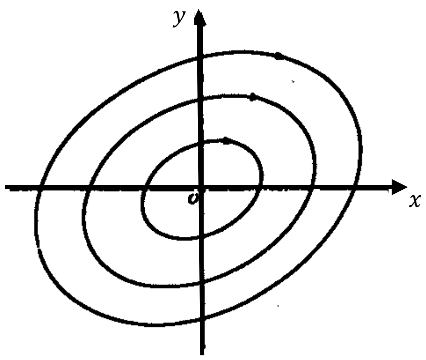


Figure 3.2

**Definition 3.2.2.** **(Saddle point)** The isolated critical point $(0, 0)$ of (3.1.4) is called a *saddle point* (Figure 3.3), if there exists Neigh-borhood of $(0, 0)$ in which the following two conditions hold:

i. There exist two paths which approach and enter $(0,0)$ from opposite directions as $t \to +\infty$ and there exist two other paths which approach and enter $(0,0)$ from different opposite directions as $t \to -\infty$.

ii. In each of the four domains between any two of the four paths in (i) there are infinitely many paths which are arbitrarily close to (0, 0) but which tend away from (0,0) both as $t \to +\infty$ and $t \to -\infty$.

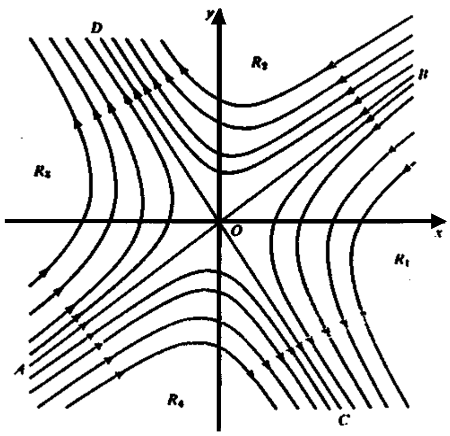


Figure 3.3

**Definition 3.2.3. (Spiral point (or focal point))** The isolated critical point $(0,0)$ of (3.1.4) is called a *spiral* (or *focal point*) (Figure 3.4), if there exists a neighborhood of $(0,0)$ such that every path $P$ in this neighborhood has the following properties:

i. P is defined for all $t > t_0$ (or for all $t < t_0$ ) for some number $t_0$ ;

ii. P approaches $(0,0)$ as $t \to +\infty$ (or as $t \to -\infty$); and

iii. P approaches $(0,0)$ in a spiral-like manner, winding around $(0,0)$ an infinite number of times as $t \to +\infty$ (or as $t \to -\infty$).

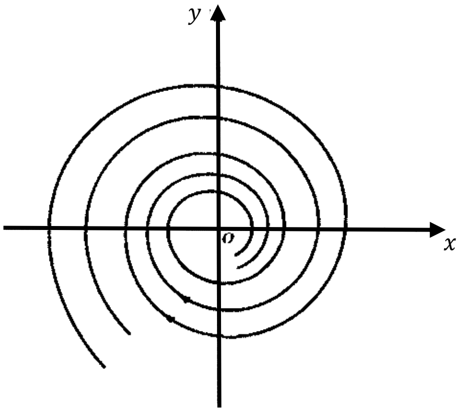


Figure 3.4

**Definition 3.2.4. (Node point)** The isolated critical point $(0, 0)$ of (3.1.4) is called *node point* (Figure 3.5), if there exist a neighborhood of (0,0) such that every path $P$ in this neighborhood has the following properties:

i. P is defined for all $t > t_0$ (or for all $t < t_0$ ) for some number $t_0$ ;
ii. P approaches $(0, 0)$ as $t \to +\infty$ (or as $t \to -\infty$); and
iii. P enters $(0, 0)$ as $t \to +\infty$ (or as $t \to -\infty$).

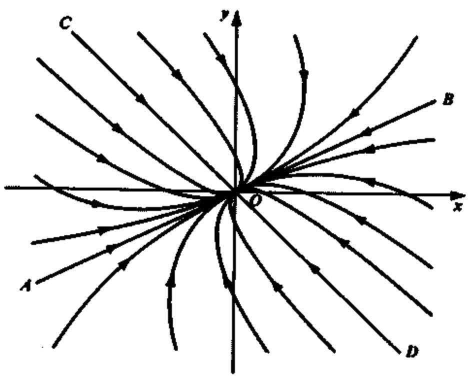


Figure 3.5

## 4. LYAPUNOV'S THEORY OF STABILITY

Since the solutions of most differential equations and systems of equations are not expressible in terms of elementary functions or quadrature, use is made of approximate methods of integration in these cases when solving concrete differential equations. The drawback of these methods lies in the fact that they yield only one particular solution; to obtain other particular solutions, one has carry out all the calculations again. Knowing one particular solutions does not permit us to draw conclusions about the character of the other solutions. In many problems of mechanics and engineering it is sometimes important to know not the specific values of a solution for some concrete value of the argument, but the type of behavior for changes in the argument and, in particular, for a boundless increase in the argument. For example, it is sometimes important to know whether the solutions that satisfy the given initial conditions are periodic, whether they approach some known function asymptotically, etc. These are the questions with which the qualitative theory of differential equations deals. One of the basic problems of the qualitative theory is that of the stability of the solution or of the stability of motion; this problem was investigated in detail by the noted Russian mathematician A. M. Lyapunov (1857-1918). ([9], [10], [12]).

Let there be given a system of differential equations:

$$\left.\begin{aligned} \frac{dx}{dt} &= f_1(t,x,y) \\ \frac{dy}{dt} &= f_2(t,x,y) \end{aligned}\right\} \tag{4.1}$$

Let $x = x(t)$ and $y = y(t)$ be the solutions of this system that satisfy the initial conditions

$$\left.\begin{aligned} x_{t=0} &= x_0 \\ y_{t=0} &= y_0 \end{aligned}\right\} \tag{4.1'}$$

Further, let $\bar{x} = \bar{x}(t)$ and $\bar{y} = \bar{y}(t)$ be the solutions of equation (4.1) that satisfy the initial conditions

$$\left.\begin{aligned} \bar{x}_{t=0} &= \bar{x}_0 \\ \bar{y}_{t=0} &= \bar{y}_0 \end{aligned}\right\} \tag{4.1''}$$

**Definition 4.1.** The solutions $x = x(t)$ and $y = y(t)$ that satisfy the equations (4.1) and the initial conditions (4.1') are called Lyapunov *stable* as $t \to \infty$ if for every arbitrarily small $\varepsilon > 0$ there is a $\delta > 0$ such that for all values $t > 0$ the following inequalities are fulfilled:

$$\left.\begin{array}{l} |\bar{x}(t) - x(t)| < \varepsilon \\ |\bar{y}(t) - y(t)| < \varepsilon \end{array}\right\} \tag{4.2}$$

if the initial data satisfy the inequalities

$$\left.\begin{array}{l} |\bar{x}_0 - x_0| < \delta \\ |\bar{y}_0 - y_0| < \delta \end{array}\right\} \tag{4.3}$$

Let us figure out the meaning of this definition. From inequalities (4.2) and (4.3) it follows that for small variations in the initial conditions, the corresponding solutions differ but little for all positive values of $t$. If the system of differential equations is a system that describes some motion, then in the case of stability of solutions, the nature of the motions changes in the initial data.

Let us analyze an example of a first order equation.

Suppose we have the differential equation:

$$\frac{dy}{dt} = -y + 1 \qquad (a)$$

The general solution of this equation is the function

$$y = Ce^{-t} + 1 \qquad (b)$$

Find a particular solution that satisfies the initial condition

$$y_{t=0} = 1 \qquad (c)$$

It is obvious that the solution $y = 1$ results when $C = 0$ (Figure 4.1).

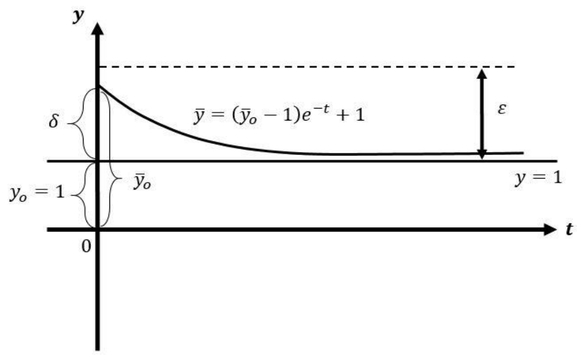


Figure 4.1

Then find the particular solution that satisfies the initial condition

$$\bar{y}_{t=0} = \bar{y}_0$$

Find the value of $C$ from equation ($b$):

$$\bar{y}_0 = C + 1$$

whence

$$C = \bar{y}_0 - 1$$

Putting this value of $C$ into equation ($b$), we get

$$\bar{y} = (\bar{y}_0 - 1)e^{-t} + 1$$

The solution $y = 1$ is obviously stable. Indeed,

$$\bar{y} - y = [(\bar{y}_0 - 1)e^{-t} + 1] - 1 = (\bar{y}_0 - 1)e^{-t} \rightarrow 0, \text{ when } t \rightarrow \infty.$$

Hence, inequality (4.3) will be fulfilled for an arbitrary $\varepsilon$ if the following inequality holds true:

$$(y_0 - 1) = \delta < \varepsilon.$$

If the equation (4.1) describe motion and the argument $t$, the time, is being given implicitly, that is, if we have a system of the form

$$\frac{dx}{dt} = f_1(x, y),$$

$$\frac{dy}{dt} = f_2(x, y).$$

then this system is termed *autonomous*.

Let us also consider the following system of linear differential equations:

$$\left.\begin{aligned} \frac{dx}{dt} &= cx + gy \\ \frac{dy}{dt} &= ax + by \end{aligned}\right\} \tag{4.4}$$

We assume that the coefficients $a, b, c, g$ are constants; by direct substitution it is clear that $x = 0$, $y = 0$ is a solution of the system (4.4). Let us investigate the question of the conditions that must be satisfied by the coefficients of the system so that the solution $x = 0$, $y = 0$ is stable. This investigation is done as follows.

Differentiate the first equation and eliminate $y$ and $\frac{dy}{dx}$ on the basis of the equations of the system

$$\frac{d^2x}{dt^2} = c\frac{dx}{dt} + g\frac{dy}{dt} = c\frac{dx}{dt} + g(ax + by) = c\frac{dx}{dt} + gax + b\left(\frac{dx}{dt} - cx\right)$$

or

$$\frac{d^2x}{dt^2} - (b + c)\frac{dx}{dt} - (ag - bc)x = 0 \tag{4.5}$$

The auxiliary equation of the differential equation (4.5) is of the form

$$\lambda^2 - (b + c)\lambda - (ag - bc) = 0 \tag{4.6}$$

This equation may be written in determinantal form:

$$\begin{vmatrix} c - \lambda & g \\ a & b - \lambda \end{vmatrix} = 0 \tag{4.7}$$

We denote the roots of the auxiliary equation (4.7) by $\lambda_1$ and $\lambda_2$.

As we shall see below, the stability or instability of the solutions of system (4.4) is determined by the nature of the roots $\lambda_1$ and $\lambda_2$ .

Let us consider all possible cases [7].

**I. The roots of the auxiliary equation are real, negative and distinct:**
$\boldsymbol{\lambda_1 < 0}$ , $\boldsymbol{\lambda_2 < 0}$ , $\boldsymbol{\lambda_1 \neq \lambda_2}$ **.**

From equation (4.5) we find $x = C_1 e^{\lambda_1 t} + C_2 e^{\lambda_2 t}$
Knowing $x$, we find $y$ from the first equation of (4.4). Thus, the solution of system (4.4) is of the form:

$$\left.\begin{aligned} x &= C_1 e^{\lambda_1 t} + C_2 e^{\lambda_2 t} \\ y &= \left[C_1(\lambda_1 - c)e^{\lambda_1 t} + C_2(\lambda_2 - c)e^{\lambda_2 t}\right]\frac{1}{g} \end{aligned}\right\} \tag{4.8}$$

**Note 4.1.** If $g = 0$ and $a \neq 0$ , then we form the form the equation (4.5) for the function $y$ . Finding $y$, we then find $x$ from the second equation of (4.4). The structure of the solution (4.8) is preserved. But if $g = 0$ , $a = 0$ , then the solution of the system of equations becomes

$$x = C_1 e^{ct} \ , \ y = C_2 e^{bt} \ . \tag{4.8$'$}$$

In this case, the analysis of the character of the solutions is easier to carry out. Choose $C_1$ and $C_2$ so that the solutions (4.8) satisfy the initial conditions

$$x\ |_{t=0} = x_0 \ , \qquad y\ |_{t=0} = y_0$$

The solution satisfying the initial conditions will be

$$\left.\begin{aligned} x &= \frac{cx_0 + gy_0 - x_0\lambda_2}{\lambda_1 - \lambda_2} e^{\lambda_1 t} + \frac{x_0\lambda_1 + cx_0 - gy_0}{\lambda_1 - \lambda_2} e^{\lambda_2 t} \\ y &= \left[\frac{cx_0 + gy_0 - x_0\lambda_2}{\lambda_1 - \lambda_2}(\lambda_1 - c)e^{\lambda_1 t} + \frac{x_0\lambda_1 - cx_0 - gy_0}{\lambda_1 - \lambda_2}(\lambda_2 - c)e^{\lambda_2 t}\right]\frac{1}{g} \end{aligned}\right\} \tag{4.9}$$

From these equations it follows that for arbitrary $\varepsilon > 0$, it is possible to select $|x_0|$ and $|y_0|$ so small that for all $t > 0$ it will be true that $|x(t)| < \varepsilon, |y(t)| < \varepsilon$ since $e^{\lambda_1 t} < 1$, $e^{\lambda_2 t} < 1$.

We note that in this case

$$\left.\begin{array}{l} \lim_{t\to+\infty} x(t) = 0 \\ \lim_{t\to+\infty} y(t) = 0 \end{array}\right\} \tag{4.10}$$

Consider the $xy-$plane. For the system of differential equations (4.4) and for the differential equation (4.5), this plane is termed the *phase plane*. We will consider the solutions (4.8) and (4.9) of system (4.4) as parametric equations of some curve in the phase plane $xOy$:

$$\left.\begin{array}{l} x = \overline{\varphi}(t, C_1, C_2) \\ y = \overline{\psi}(t, C_1, C_2) \end{array}\right\} \tag{4.11}$$

$$\left.\begin{array}{l} x = \varphi(t, x_0, y_0) \\ y = \psi(t, x_0, y_0) \end{array}\right\} \tag{4.12}$$

These curves are the *integral curves (solution curves)* of the differential equation

$$\frac{dy}{dx} = \frac{ax+by}{cx+gy} \tag{4.13}$$

Which is obtained from the system (4.4) by dividing the right and left members by each other. The origin, $O(0,0)$ is a *singular point* of the differential equation (4.13), since this point does not belong to the domain of existence and uniqueness of the solution. The nature of the solution (4.9) and, generally, of the solutions of the system (4.4) is illustrated by the arrangement of the integral curves

$$\bar{F}(x, y, C) = 0$$

Which form the complete integral of the differential equation (4.13). The constant $C$ is determined from the initial condition $y_{x=x_0} = y_0$. Substituting the value of $C$, we obtain the equation of the family in the form

$$F = (x, y, x_0, y_0) \tag{4.14}$$

In the case of solutions (4.9), the singular point is called a *stable nodal point*. We say that a point moving along the integral curve approaches a singular point without bound as $t \to +\infty$.

It is obvious that the relation (4.14) may be obtained by eliminating the parameter $t$ from the system (4.12). We will not continue the analysis of the argument of integral curves near a singular point on the phase plane for all possible cases of roots of the auxiliary equation and will confine ourselves to an illustration of this fact in elementary instances that do not require unwieldy computations. We note that for arbitrary coefficients the behavior of integral curves of the equation (4.13) near the origin is qualitatively the as is now to be examined in the following example.

**Example 4.1.** Investigate the stability of the solution $x = 0$, $y = 0$ of the system equations

$$\frac{dx}{dt} = -x,$$

$$\frac{dy}{dt} = -2y.$$

**Solution:** The auxiliary equation is

$$\begin{vmatrix} -1-\lambda & 0 \\ 0 & -2-\lambda \end{vmatrix} = 0$$

The roots of the auxiliary equation are $\lambda_1 = -1$, $\lambda_2 = -2$

In this case, the solutions (4.8′) will be $x = C_1 e^{-t},\ y = C_2 e^{-2t}$

The solutions (4.9) will be $$x = x_0 e^{-t},\ y = y_0 e^{-2t} \qquad (a)$$

Clearly, $x(t) \to 0$ and $y(t) \to 0$ as $t \to +\infty$. The solution $x = 0, y = 0$ is stable. Now let us examine the phase plane. Eliminating the parameter $t$ from the equation ($a$), we get an equation of type (4.14).

$$\left(\frac{x}{x_0}\right)^2 = \frac{y}{y_0} \qquad (b)$$

This is a family of parabolas (Figure 4.2)

The equation of type (4.13) will, for the given example, be $\frac{dy}{dx} = \frac{2y}{x}$

Integrating, we get

$$ln|y| = 2ln|x| + ln|C| \quad \Longrightarrow \quad y = Cx^2 \qquad (c)$$

Determine $C$ from the condition

$$y_{x=x_0} = y_0\,, \;\; C = \frac{y_0}{{x_0}^2}\;.$$

Substituting the value of $C$ thus found into ($c$), we get the solution ($b$). The singular point $O\ (0,0)$ is a *stable nodal point*.

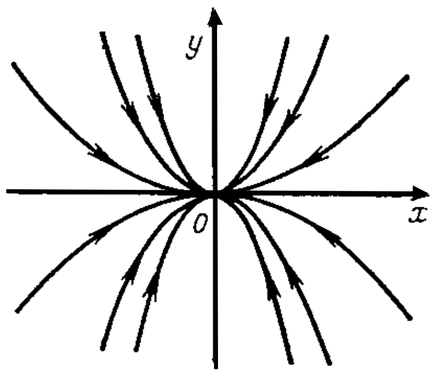


Figure 4.2

II. **The roots of the auxiliary equation are real, positive and distinct:**

$\boldsymbol{\lambda_1 > 0\,,\ \lambda_2 > 0\,, \lambda_1 \neq \lambda_2\,.}$

In this case the solutions are also expressed by the formulas (4.8) and (4.9). But in this case, for arbitrarily small $|x_0|$ and $|y_0|$ it will be true that $|x(t)| \to \infty$, $|y(t)| \to \infty$ as $t \to +\infty$, since $e^{\lambda_1 t} \to \infty$ and $e^{\lambda_2 t} \to \infty$ as $t \to +\infty$. On the phase plane, the singular point is ad stable nodal point: as

$t \to +\infty$ the point on the integral curve recedes from the rest point $x = 0\,,\ y = 0$.

**Example 4.2.** Investigate the stability of the solutions of the system

$$\frac{dx}{dt} = x,$$

$$\frac{dy}{dt} = 2y.$$

**Solution:** The auxiliary equation is

$$\begin{vmatrix} 1-\lambda & 0 \\ 0 & 2-\lambda \end{vmatrix} = 0$$

The roots of the auxiliary equation are $\lambda_1 = 1\,,\ \lambda_2 = 2$

The solutions will be $x = x_0 e^t\,,\ \ y = y_0 e^{2t}$ $(a)$

The solution is unstable, since $|x(t)| \to \infty\ ,\ |y(t)| \to \infty$ as $t \to +\infty$.

Eliminating the parameter $t$, we obtain

$$\left(\frac{x}{x_0}\right)^2 = \frac{y}{y_0} \qquad (b)$$

(Figure 4.3) The singular point $O\ (0,0)$ is an *unstable nodal point*.

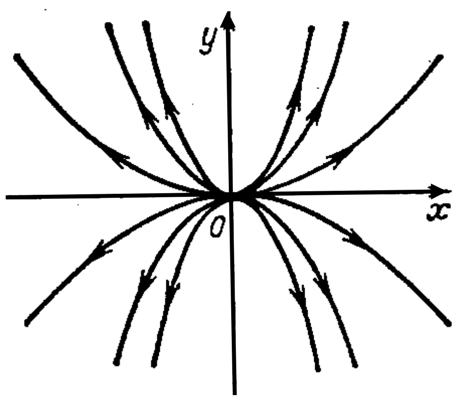


Figure 4.3

**III.** **The roots of the auxiliary equation are real and unlike sign, for example:** $\boldsymbol{\lambda_1 > 0}$, $\boldsymbol{\lambda_2 < 0}$.

From formulas (4.9) it follows that for arbitrarily small $|x_0|$ and $|y_0|$,

If $cx_0 + gy_0 - x_0\lambda_2 \neq 0$, it will be true $|x(t)| \to \infty$, $|y(t)| \to \infty$ as $t \to +\infty$. The solution is unstable. The singular point on the phase plane is termed a *saddle point*.

**Example 4.3.** Investigate the stability of the solutions of the system

$$\frac{dx}{dt} = x$$

$$\frac{dy}{dt} = -2y$$

**Solution:** The auxiliary equation is

$$\begin{vmatrix} 1-\lambda & 0 \\ 0 & -2-\lambda \end{vmatrix} = 0$$

The roots of the auxiliary equation are $\lambda_1 = 1$, $\lambda_2 = -2$

The solution is $x = x_0 e^t$, $y = y_0 e^{-2t}$

The solution is unstable. Eliminating the parameter $t$, we get a family of curves on the phase plane: obtain

$$yx^2 = y_0\ {x_0}^2$$

The singular point $O$ (0,0) is an *unstable saddle point* (Figure 4.4).

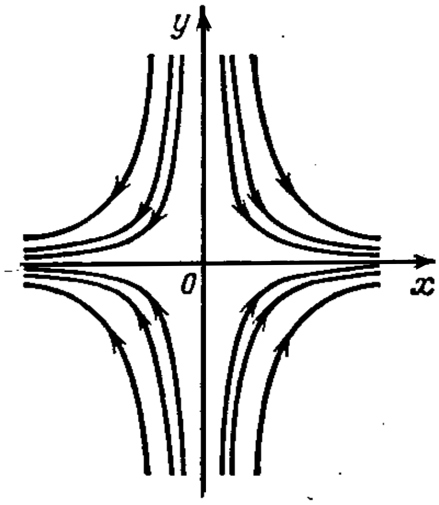


Figure 4.4

**IV. The roots of the auxiliary equation are complex with negative real part:**
$\boldsymbol{\lambda_1 = \alpha + i\beta}$, $\boldsymbol{\lambda_2 = \alpha - i\beta}$ $(\boldsymbol{\alpha < 0})$.

The solution of system (4.4) is

$$\left.\begin{aligned} x &= e^{at}[C_1\, cos\beta t + C_2 \sin\beta t] \\ y &= \frac{1}{g} e^{at}[(\alpha C_1 + \beta C_2 - cC_1) cos\beta t + (\alpha C_2 - \beta C_1 - cC_2)\sin\beta t] \end{aligned}\right\} \tag{4.15}$$

Introducing the notation

$$C = \sqrt{C_1^2 + C_2^2}\,, \quad \sin\delta = \frac{C_1}{C}\,, \quad \cos\delta = \frac{C_2}{C}$$

We can rewrite equations (4.15) as

$$\left.\begin{aligned} x &= Ce^{at}\sin(\beta t + \delta) \\ y &= \frac{Ce^{at}}{g}\,[(\alpha - c)\sin(\beta t + \delta) + \beta\cos(\beta t + \delta)] \end{aligned}\right\} \tag{4.16}$$

Where $C_1$ and $C_2$ are arbitrary constants that are determined from the initial conditions:
$x = x_0\,, y = y_0$, when $t = 0$, and $x_0 = C\sin\delta\,, y_0 = \frac{C}{g}\,[(\alpha - c)\sin\delta + \beta\cos\delta]$

whence we find

$$C_1 = x_0\,, \qquad C_2 = \frac{gy_0 - x_0(\alpha - c)}{\beta} \tag{4.17}$$

Note again that if $g = 0$, then the form of the solution will be somewhat different, but the nature of the analysis doesn't change.

It is obvious that for arbitrary $(\varepsilon > 0)$. Given sufficiently small $|x_0|$ and $|y_0|$, the following relation will hold:

$$|x(t)| < \varepsilon\,, \quad |x(t)| < \varepsilon$$

The solution is *stable*. In this case, as $t \to +\infty$, $x(t) \to 0$ and $y(t) \to 0$

changing sign an unlimited number of times. The singular point on the phase plane is called a *stable focal point.*

**Example 4.4.** Investigate the stability of the solution of the system

$$\frac{dx}{dt} = -x + y,$$

$$\frac{dy}{dt} = -x - y.$$

**Solution:** from the auxiliary equation and find its roots:

$$\begin{vmatrix} -1-\lambda & 1 \\ -1 & -1-\lambda \end{vmatrix} = 0\,, \quad \lambda^2 + 2\lambda + 2 = 0$$

$$\lambda_{1,2} = -1 \pm i\,, \qquad \alpha = -1\,, \qquad \beta = 1$$

We find $C_1$ and $C_2$ from formulas (4.17): $C_1 = x_0\,,\quad C_2 = y_0\,.$

Substituting into (4.15), we get

$$\left.\begin{array}{l} x = e^{-t}(x_0 \cos t + y_0 \sin t) \\ y = e^{-t}(y_0 \cos t - x_0 \sin t) \end{array}\right\} \quad (A)$$

It is obvious that for arbitrary values of $t$ , $|x| \leq |x_0| + |y_0|$ , $|y| \leq |x_0| + |y_0|$ as $t \to +\infty$ , $x(t) \to 0$ and $y(t) \to 0$, and the solution is *stable*.

Let us analyze the arrangement of the curves on the phase plane in this case. We transform expressions ($A$).

Let $x_0 = M \cos \delta$ , $y_0 = M \sin \delta$ , $M = \sqrt{{x_0}^2 + {y_0}^2}$ , $\tan \delta = \frac{y_0}{x_0}$

Then equation ($A$) become

$$\left.\begin{array}{l} x = Me^{-t} \cos(\beta t - \delta) \\ y = Me^{-t} \sin(\beta t - \delta) \end{array}\right\} \quad (B)$$

Pass to polar coordinates $\rho$ and $\theta$ in the phase plane and establish the relationship $\rho = f(\theta)$. Equation ($B$) become

$$\left.\begin{array}{l} \rho \cos \theta = Me^{-t} \cos(\beta t - \delta) \\ \rho \sin \theta = Me^{-t} \sin(\beta t - \delta) \end{array}\right\} \quad (C)$$

Squaring the right and left members and adding, we obtain

$$\rho^2 = M^2 e^{-2t}$$

or

$$\rho = Me^{-t} \quad (D)$$

How does $t$ depend on $\theta$ , dividing the terms of the lower equation of ($C$) by the corresponding terms of the upper equation, we get

$$\tan \theta = \tan(\beta t - \delta)$$

whence

$$t = \frac{\theta + \delta}{\beta}$$

Substituting into ($D$), we have

$$\rho = Me^{-\frac{\theta+\delta}{\beta}}$$

or

$$\rho = Me^{-\frac{\delta}{\beta}-\frac{\theta}{\beta}}$$

Putting $Me^{-\frac{\delta}{\beta}} = M_1$, we finally obtain

$$\rho = M_1 e^{-\frac{\theta}{\beta}} \qquad (E)$$

This is a family of logarithmic spirals. In this case, a point moving along an integral curve approaches the origin as $t \to \infty$. The singular point $O(0,0)$ is a *stable focal point*.

V. **The roots of the auxiliary equation are complex with positive real part: $\boldsymbol{\lambda_1 = \alpha + i\beta}$, $\boldsymbol{\lambda_2 = \alpha - i\beta}$ $(\boldsymbol{\alpha > 0})$.**

Here the solution will also be expressed by the formulas (4.15), where $\alpha > 0$ for arbitrary initial conditions $x_0$ and $y_0$ $\left(\sqrt{x_0^2 + y_0^2} \neq 0\right)$, $|x(t)|$ and $|y(t)|$ can assume arbitrary large value as $t \to +\infty$. The solution is *unstable*. The singular point in the phase plane is called an *unstable focal point*. A point on an integral curve recedes without bound from the origin of coordinates.

**Example 4.5.** Investigate the stability of the solution of the system of equations.

$$\frac{dx}{dt} = x + y,$$

$$\frac{dy}{dt} = -x + y.$$

**Solution:** Form the auxiliary equation

$$\begin{vmatrix} 1-\lambda & 1 \\ -1 & 1-\lambda \end{vmatrix} = 0, \quad \lambda^2 - 2\lambda + 2 = 0$$

$$\lambda_1 = 1 + i, \quad \lambda_2 = 1 - i$$

Taking into account (4.17), the solution (4.15) in this case will be

$$x = e^{t}(x_0 cost + \ y_0 \sin t)$$

$$y = e^{t}(y_0 cost - \ x_0 \sin t)$$

In the phase plane, we obtain the curve in polar coordinates:

$$\rho = \ \bar{M}_1 e^{\theta/\beta}$$

The singular point is an *unstable focal point.* (Figure 4.5).

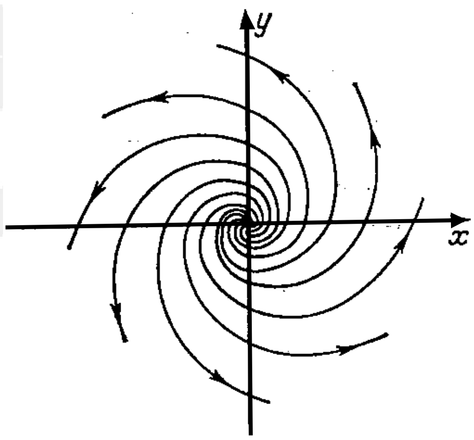


Figure 4.5

**VI. The roots of the auxiliary equation are pure imaginary:**

$\boldsymbol{\lambda_1 = i\beta}$ , $\boldsymbol{\lambda_2 = -i\beta}$ **.**

The solution (4.15) in this case assume the form

$$\left.\begin{aligned} x &= C_1 \, cos\beta t + \ C_2 \sin \beta t \\ y &= \frac{1}{g} \, [(\beta C_2 - cC_1) cos\beta t + (-\beta C_1 - cC_2) \sin \beta t] \end{aligned}\right\} \qquad (4.18)$$

The constant $C_1$ and $C_2$ are found from formulas (4.17):

$$C_1 = x_0\,, \quad C_2 = \frac{gy_0 + cx_0}{g} \tag{4.19}$$

Clearly, for arbitrary $\varepsilon > 0$ and for all sufficiently small $|x_0|$ and $|y_0|$ it will be true that $|x(t)| < \varepsilon\,,\ |y(t)| < \varepsilon$ for arbitrary $t$. The solution is *stable*. Here $x$ and $y$ are periodic functions of $t$. In order to carry out the analysis of the integral curves on the phase plane, it is advisable to write the first equation of (4.18) as see (4.16).

$$\left.\begin{aligned} x &= C\sin(\beta t + \delta) \\ y &= \frac{C\beta}{g}\cos(\beta t + \delta)\,\beta t\ -\ \frac{Cc}{g}\sin(\beta t + \delta) \end{aligned}\right\} \tag{4.20}$$

where $C$ and $\delta$ are arbitrary constants.

From the expressions (4.20) it follows that $x$ and $y$ are periodic function of $t$. Eliminate the parameter $t$ from (4.20):

$$y = \frac{C\beta}{g}\sqrt{1 - \frac{x^2}{C^2}} - \frac{c}{g}x$$

Eliminating the radical, we get

$$\left(y - \frac{c}{g}x\right)^2 = \left(\frac{C\beta}{g}\right)^2\left(1 - \frac{x^2}{C^2}\right) \tag{4.21}$$

This is a family of quadric curves (which are real) that depend on an arbitrary constant $C$. Neither of them *recedes to infinity*. Consequently, this is a family of ellipses around the origin (for $c = 0$, the axes of the ellipses are parallel to the coordinate axes). The singular point termed the *Centre* (Figure 4.6).

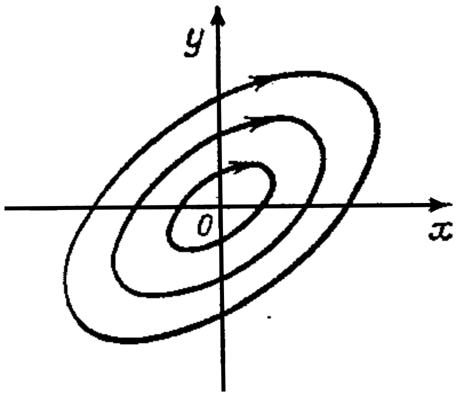


Figure 4.6

**Example 4.6.** Investigate the stability of the solution of the system of equations

$$\frac{dx}{dt} = y\,, \quad \frac{dy}{dt} = -4x.$$

**Solution:** From the auxiliary equation and find its roots:

$$\begin{vmatrix} -\lambda & 1 \\ -4 & -\lambda \end{vmatrix} = 0\,, \quad \lambda^2 - 4 = 0\,, \quad \lambda_{1,2} = \pm 2i$$

The solutions (4.20) will be

$$x = C\sin(2t + \delta)\,,$$

$$y = 2C\cos(2t + \delta)$$

Equation (4.21) will assume the form

$$y^2 = 4C^2\left(1 - \frac{x^2}{C^2}\right), \quad \frac{y^2}{4C^2} + \frac{x^2}{C^2} = 1.$$

We have a system of ellipses on the phase plane, and the singular point is a *center*.

**VII. Let $\lambda_1 = 0$, $\lambda_2 < 0$.**

The solution (4.8) in this case becomes

$$\left.\begin{aligned} x &= C_1 + C_2 e^{\lambda_2 t} \\ y &= \frac{1}{g}\left[-C_1 c + C_2(\lambda_2 - c)e^{\lambda_2 t}\right] \end{aligned}\right\} \tag{4.22}$$

Clearly, for arbitrary $\varepsilon > 0$ and for sufficiently small $|x_0|$ and $|y_0|$ it will be true that $|x(t)| < \varepsilon$, $|y(t)| < \varepsilon$ when $t > 0$. Hence, the solution is *stable.*

**Example 4.7.** Investigate the stability of the solution of the system

$$\frac{dx}{dt} = 0\,, \quad \frac{dy}{dt} = -y. \tag{$\alpha$}$$

**Solution:** We find the roots of the auxiliary equation

$$\begin{vmatrix} -\lambda & 0 \\ 0 & -1-\lambda \end{vmatrix} = 0\,, \quad \lambda^2 + \lambda = 0\,, \quad \lambda_1 = 0\,, \quad \lambda_2 = -1$$

Here, $g = 0$. The solution are found directly by solving the system and without using the formulas (4.22):

$$x = C_1\,, \quad y = C_2 e^{-t} \tag{$\beta$}$$

The solution satisfying the initial conditions

$$x = x_0\,, \quad y = y_0 e^{-t} \tag{$\gamma$}$$

The solution is clearly *stable*.

The differential equation on the phase plane will be $\frac{dx}{dy} = 0$. The complete integral will be $x = C$. The integral curves are straight lines parallel to the $y$-axis from the equation $(\gamma)$ it follows that the point along the integral curves approach the straight line $y = 0$ (Figure 4.7).

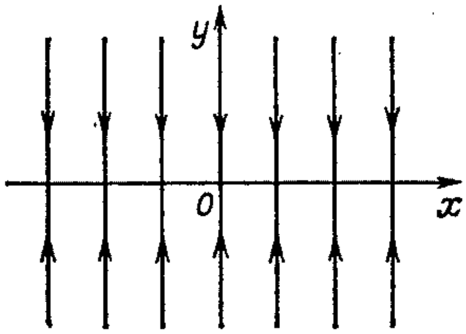


Figure 4.7

**VIII. Let $\lambda_1 = 0$ , $\lambda_2 > 0$.**

From formulas (4.22) or (4.8') it follows that the solution is unstable since $|x(t)| + |y(t)| \to \infty$ as $t \to +\infty$.

**IX. Let $\lambda_1 = \lambda_2 < 0$.**

The solution is

$$\left.\begin{aligned} x &= (C_1 + C_2 t)e^{\lambda_1 t} \\ y &= \frac{1}{g}\, e^{\lambda_1 t}[C_1(\lambda_1 - c) + C_2(1 + \lambda_1 - ct)] \end{aligned}\right\} \tag{4.23}$$

Since $e^{\lambda_1 t} \to 0$ and $te^{\lambda_1 t} \to 0$ , then for an arbitrary $\varepsilon > 0$ it is possible to choose $C_1$ and $C_1$ (by selecting $x_0$ and $x_0$ ) such that it will be true that that $|x(t)| < \varepsilon$ , $|y(t)| < \varepsilon$ , for arbitrary $t > 0$. The solution is thus stable, and $x(t) \to 0$ and $y(t) \to 0$ as $t \to +\infty$.

**Example 4.8.** Investigate the stability of the solution of the system

$$\frac{dx}{dt} = -x ,$$

$$\frac{dy}{dt} = -y .$$

**Solution:** We find the roots of the auxiliary equation:

$$\begin{vmatrix} -1-\lambda & 0 \\ 0 & -1-\lambda \end{vmatrix} = 0\,, \quad (\lambda+1)^2 = 0\ , \quad \lambda_1 = \lambda_2 = -1$$

Here $g = 0$. The solution of the system will be of the form (4.8′):

$$x = C_1 e^{-t}\,,\ \ y = C_2 e^{-t}$$

and $x \to 0\,,\ y \to 0$ as $t \to +\infty$. The solution is stable. The family of curves on the phase plane will be

$$\frac{y}{x} = \frac{C_2}{C_1} = k\,, \text{ that is, } y = kx$$

This is a family of straight lines passing through the origin. The points along the integral curves approach the origin. The singular point $O(0,0)$ is a *nodal point* (Figure 4.8).

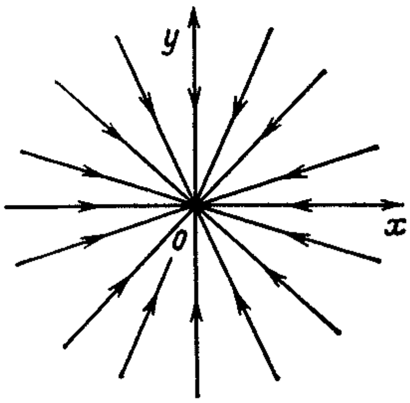


Figure 4.8

Note that in the case of $\lambda_1 = \lambda_2 < 0$ the form of the solution (4.22) is preserved, but when $t \to +\infty\,,\ |x(t)| \to \infty$ and $|y(t)| \to \infty$ . The solution is *unstable*.

**X.** **Let $\boldsymbol{\lambda_1 = \lambda_2 = 0}$ .** Then

$$\left.\begin{aligned} x &= C_1 + C_2 t \\ y &= \frac{1}{g}\left[-cC + C_2 - cC_2 t\right] \end{aligned}\right\} \tag{4.24}$$

. Whence it is quite clear that $x \to \infty$ and $y \to \infty$ as $t \to +\infty$ the solution is *unstable*.

**Example 4.9.** Investigate the stability of the solution of the system of equations

$$\frac{dx}{dt} = y\,, \qquad \frac{dy}{dt} = 0.$$

**Solution:** Find the roots of the auxiliary equation

$$\begin{vmatrix} -\lambda & 1 \\ 0 & -\lambda \end{vmatrix} = 0\,, \quad {\lambda_1}^2 = 0\,, \quad \lambda_1 = \lambda_2 = 0$$

We find the solutions to be

$$y = C_2\,, \quad x = C_2 t + C_1$$

It is quite obvious that $x \to \infty$ as $t \to +\infty$. The solution is *unstable*. The equation on the phase plane is $\frac{dy}{dx} = 0$. The integral curves $y = C$ are straight lines parallel to the axis (Figure 4.9). The singular point is called a *degenerate saddle point*.

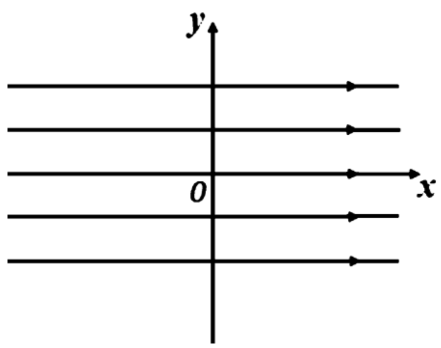


Figure 4.9

To give a general criterion of the stability of solution of the system (4.4), we do as follows. We write the roots of the auxiliary equation in the form of complex number:

$$\lambda_1 = \lambda_1^* + i\lambda_1^{**}$$

$$\lambda_2 = \lambda_2^* + i\lambda_2^{**}$$

(In the case of real roots, $\lambda_1^{**} = 0$ and $\lambda_2^{**} = 0$).

Let us take the plane of a complex variable $\lambda^* \lambda^{**}$ and display the roots of the auxiliary equation by points in this plane. Then, on the basis of the cases that have been considered, the condition of stability of solution of the system (4.4) may be formulated as follows.

If not a single one of the roots $\lambda_1$, $\lambda_2$ of the auxiliary equation (4.6) lies to the right of the axis of imaginaries, and at least one root is nonzero, then the solution is stable; if at least one root lies to the right axis of imaginaries, or both roots are equal to zero, then the solution is *unstable* (Figure 4.10).

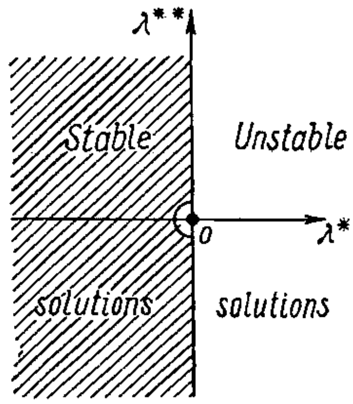


Figure 4.10

Let us now consider a more general system of equations;

$$\left.\begin{aligned} \frac{dx}{dt} &= cx + gy + P(x,y) \\ \frac{dy}{dt} &= ax + by + Q(x,y) \end{aligned}\right\} \qquad (4.25)$$

But for exceptional cases, the solution of this system is not expressible in terms of elementary function and quadrature.

To establish whether the solution of this system are stable or unstable, they are compared with the solutions of a linear system. Suppose that for $x > 0$ and $y = 0$, the functions $P\,(x,y)$ and $Q\,(x,y)$ also approach zero and approach it faster than $\rho$, where

$\rho = \sqrt{x^2 + y^2}$; in other words,

$$\lim_{\rho \to 0} \frac{P(x,y)}{\rho} = 0 \; ; \qquad \lim_{\rho \to 0} \frac{Q(x,y)}{\rho} = 0$$

Then it may be proved that, save for the exception case, the solution of the system (4.25) will be stable when the solution of the system

$$\left.\begin{aligned} \frac{dx}{dt} &= cx + gy \\ \frac{dy}{dt} &= ax + by \end{aligned}\right\} \tag{4.4}$$

is stable, and unstable when the solution of the system (4.4) is unstable. The exception is that case when both roots of the auxiliary equation lie on the axis of imaginaries; then the question of the stability or instability of solution of the system (4.25) is considerably more involved.

Lyapunov [10] investigated the question of the stability of solutions of system of equation for rather general assumptions concerning the form of these equations.

In oscillation theory, one often has to deal with the equation

$$\frac{d^2x}{dt^2} = f\left(x, \ \frac{dx}{dt}\right) \tag{4.26}$$

Put

$$\frac{dx}{dt} = v \tag{4.27}$$

Then we get the system of equations

$$\left.\begin{aligned} \frac{dx}{dt} &= v \\ \frac{dv}{dt} &= f(x, v) \end{aligned}\right\} \tag{4.28}$$

The phase plane for this system is the *xv*-plane. The integral curves on the phase plane geometrically represent the velocity *v* as a function of the *x*-coordinate and give a pictorial and qualitative description of the variation of $x$ and $v$. If the point $x = 0$, $v = 0$ is a singular point, then it determines a position of equilibrium.

Thus, for example, if the singular of a system of equations is a center that is the integral curves on the phase plane are closed curves circling the origin, then the motion described by equation (4.26) are undamped oscillations. If the singular point of the phase plane is a focal point (and then $|x| \to 0$ , $|v| \to 0 \;\; as \;\; t \to \infty$ ), then the motion defined by equation (4.26) are damped oscillations. If the singular point is a nodal point or a saddle point (and this is the only singular point), then $x \to \pm\infty \;\; as \;\; t \to \infty$ . In this case a moving material point recedes to infinity.u

If equation (4.26) is a linear equation of the form $\frac{d^2x}{dt^2} = ax + b\,\frac{dx}{dt}$ , then the system (4.28) looks like

$$\frac{dx}{dt} = v \;\; , \;\; \frac{dv}{dt} = ax + bv.$$

This is a system of type (4.4). The point $x = 0$ , $v = 0$ is a singular point, it defines a position of equilibrium. Note that variable $x$ is not necessarily a mechanical displacement of a point. It may have a variety of physical meanings; for instance, it may represent a quantity describing electrical oscillations.

# 5. APPLICATION IN BIOLOGY

The conjoining of mathematics and biology has brought about significant advances in both areas, with mathematics providing a tool for modelling and understanding biological phenomena and biology stimulating developments in the theory of nonlinear differential equations. The continued application of mathematics to biology holds great promise and in fact may be the applied mathematics of the $21^{th}$ century.

Provides a detailed treatment of ordinary differential equations, techniques for their solution, and their use in biological applications. The presentation includes the fundamental techniques of nonlinear differential equations, biological modelling, chemical Kinetics and the SEIR model. ([14], [15], [16], [17], [18]).

## 5.1 Mathematical modeling

A mathematical modeling is a mathematical object based on a real situation and created in the hope that its mathematical behavior resembles the real behavior. Mathematical modeling is the art or science of creating, analyzing, validating, and interpreting mathematical models. In other words, mathematical modeling is an area of applied mathematical concerned with describing and predicting real-world system behavior. There are some examples of real-world systems: an object moving in a gravitational field; stock market fluctuations; predator-prey interactions; cell signaling pathways. Models are abstractions of reality, there are a representation of a particular thing, idea, or condition. Mathematical models are characterized by assumption about:

- Variables (the things which change).
- Parameters (the things which do not change).
- Functional forms (the relationship between the two).

## 5.2 The modeling process

The modeling process is a series of steps taken to convert an idea first in to a conceptual (theoretical) model and then into a quantitative model. A conceptual model presents our ideas about how the system works. It is expressed visually in a model diagram, typically involving boxes (state variables) and arrows (material flows or causal effects). Equations are developed for the rates of each process and are combined to form a quantitative model consisting a dynamic (i.e., varying with time) equations for each state variable. The dynamic equations can then be studied mathematically or translated into computer code to obtain numerical solutions for state variable trajectories.

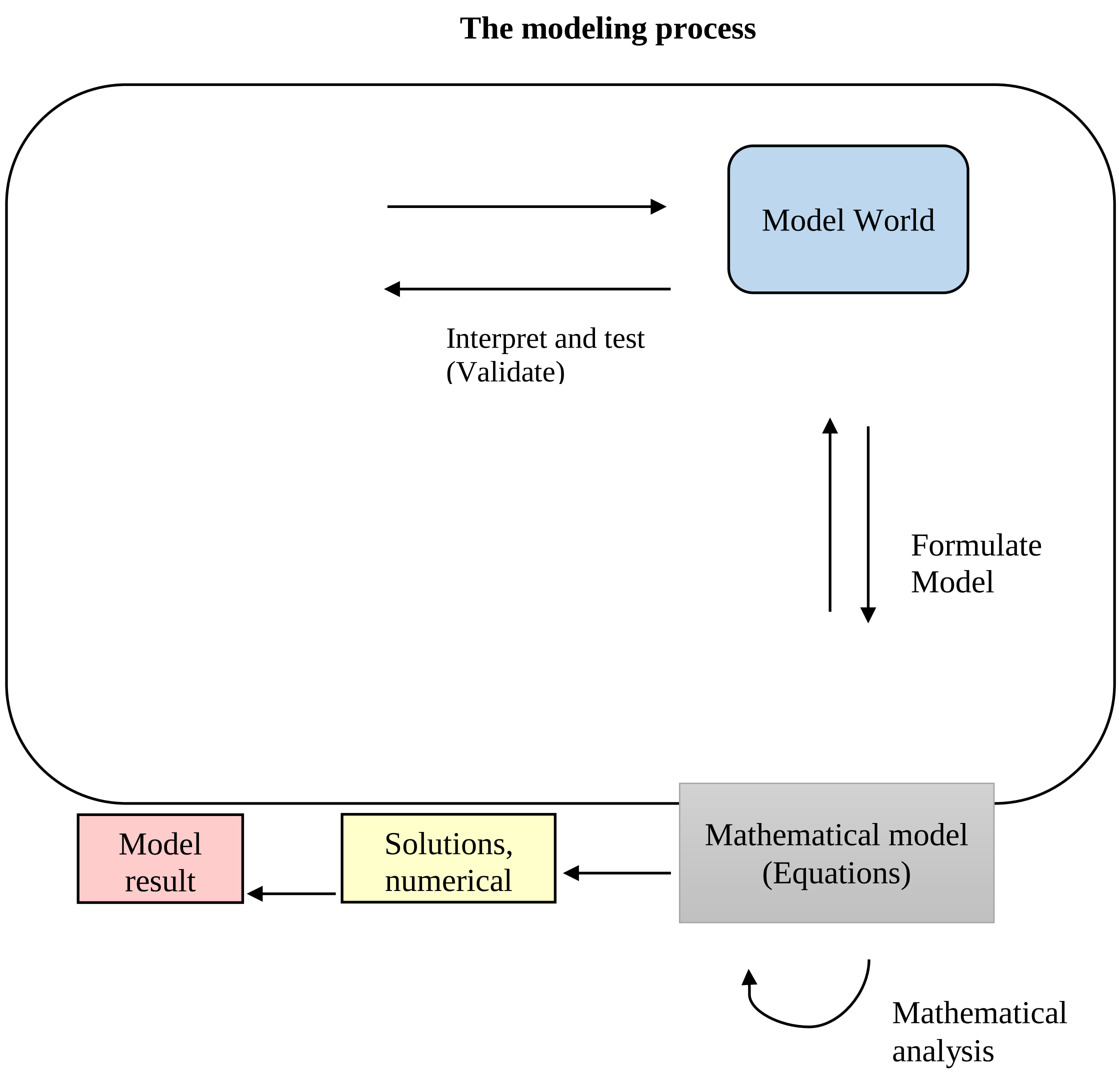


Figure 5.1 (Mathematical modelling steps)

### 5.3 Chemical Kinetics

The classical theory of chemical kinetics is used to formally represent biological processes for mathematical modeling. The assumption is that a model consists of:

- A vector of components (species) $A = (A_1, A_2, \ldots., A_m)$, for each component $A_j$, $j = 1, 2, \ldots, m$ a non negative variable $c_j$ (concentration of $A_j$, i.e. $c_j = [\ A_j\ ]$) is defined: the vector of concentrations is $C$. In other words, $c_j = N_i/V$ where $V$ is volume, $N_i$ is non-negative real extensive variable (the number of molecules of that species).
- A vector of reactions $V = (v_1, v_2, \ldots., v_n)$.
- A vector of kinetic constants $K = ({k_1}^{\mp}, {k_2}^{\mp}, \ldots., {k_n}^{\mp})$. The kinetic constants depend on reaction conditions (e.g. temperature, ph., solvent, etc.)

For the following $n$ reversible reactions which are represented by its stoichiometric equations

$$\sum_{j=1}^{m} \alpha_{ij} A_{ij} \underset{k_i^-}{\overset{k_i^+}{\Leftrightarrow}} \sum_{j=1}^{m} \beta_{ij} A_j \ , \quad i = 1,2, \ldots., n\ . \tag{5.3.1}$$

The non-negative integers $\alpha_{ij}$ and $\beta_{ij}$ are called stoichiometric coefficients. The standard mass action law is used to define the rate of reactions. The reaction rates are:

$$v_i = k_i^+ \prod_{j=1}^{m} c_j^{\alpha_{ij}}\ (t) - k_i^- \prod_{j=1}^{m} c_j^{\ \beta_{ij}}\ (t)\ , \quad i = 1, 2, \ldots., n \tag{5.3.2}$$

Where ( $k_i^+ > 0$ ) and ($k_i^- \geq 0$ ) are the reaction rate coefficients. [14], [15], [16].

The stoichiometric matrix is $\varsigma = (\gamma_{ij})$, where $\gamma_{ij} = \beta_{ij} - \alpha_{ij}$, for $i = 1, 2, \ldots.,\ n$ and $j = 1, 2,\ \ldots., m$. The stoichiometric vector $\gamma_i$ is the $i$th row of $\varsigma$ with coordination $\gamma_{ij} = \beta_{ij} - \alpha_{ij}$, [17].

Let us give an example of stoichiometric vector, for a nonlinear chain of chemical reactions ( $A_1 + 2A_2 \rightarrow 2A_3 \rightarrow A_4 + A_5 \rightarrow A_6$ ), we have three stoichiometric vectors as follows:

$$\gamma_1 = \begin{pmatrix} -1 \\ -2 \\ 2 \\ 0 \\ 0 \\ 0 \end{pmatrix} \ . \ \gamma_2 = \begin{pmatrix} 0 \\ 0 \\ -2 \\ 1 \\ 1 \\ 0 \end{pmatrix} \ , \ \gamma_3 = \begin{pmatrix} 0 \\ 0 \\ 0 \\ -1 \\ -1 \\ 1 \end{pmatrix}$$

The system of ordinary differential equations describes the dynamics of chemical reactions. The kinetic equations are:

$$\frac{dC}{dt} = W(C(t), K) = \varsigma\, v(C(t), K), \qquad (5.3.3)$$

$$C(0) = C_0\,, \qquad t \in I \subset \mathbb{R}^+ \cup \{0\}$$

Where $\varsigma$ a stoichiometric matrix of m by is $n, C\ (0)$ is a vector of initial concentrations.

The equation of stoichiometric conservation law is given:

$$\sum_j b_j c_j = B \qquad (5.3.4)$$

where $B$ is a constant

The differential for a particular component $A^*$ in a model is written as:

$$\frac{dc_{A^*}}{dt} = \sum \beta \ldots v_{A^*,produced} - \sum \alpha \ldots v_{A^*,consumed} \qquad (5.3.5)$$

This means (rate of the change of $c_{A^*}$ ) =
(concentration of A* formed in all reactions) – (concentration of A* consumed in all reactions)

Where $v_{A^*}$ is the rate of formation/consumption of species $A^*$ in a particular reaction, [18].

### 5.4 The SEIR model

The SEIR model with nonlinear incidence rates in epidemiology is studied. Global stability of the endemic equilibrium is proved using a general criterion for the orbital stability of periodic orbits associated with higher-dimensional nonlinear autonomous systems as well as the theory of competitive systems of differential equations.
We'll now consider the epidemic model in which the population consists of four groups:

- $S$ is the fraction of susceptible individuals (those able to contract the disease),
- $E$ is the fraction of exposed individuals (those who have been infected but are not yet infectious),
- $I$ is the fraction of infective individuals (those capable of transmitting the disease),
- $R$ is the fraction of recovered individuals (those who have become immune).

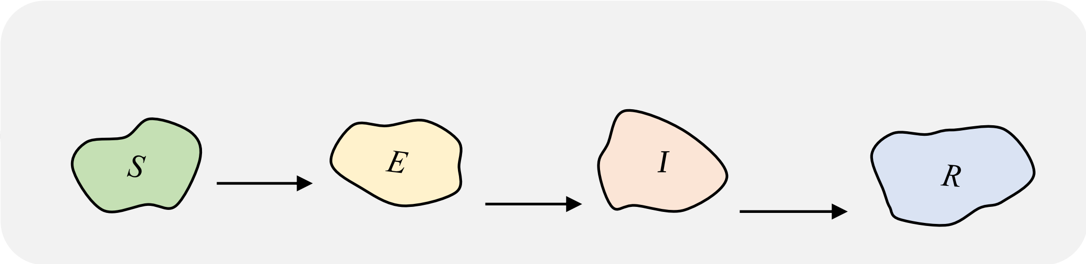


Figure 5.2

**Note 5.4.1.** The variables give the *fraction* of individuals - that is, we have normalized them so that

$$S + E + I + R = 1$$

The model has five parameters as follows:

| | |
|---|---|
| $\beta$ | is the infection rate |
| $\alpha$ | is the immunity loss rate |
| $\gamma$ | is the infectious recovery rate |
| $\varepsilon$ | is the incubation rate |
| $P$ | is the time varying total population |

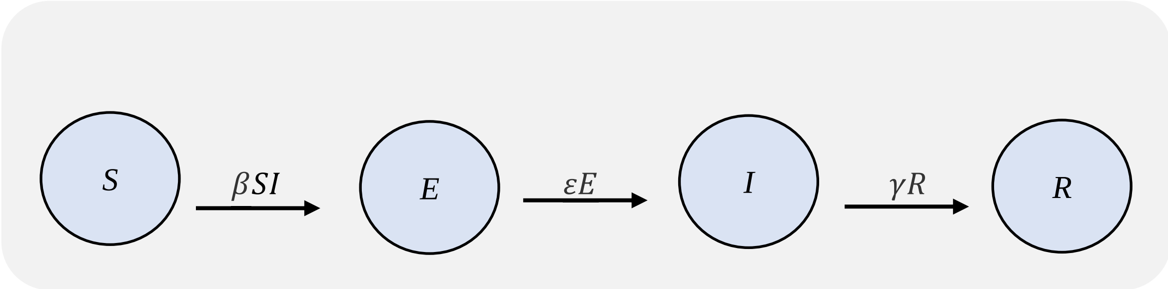


Figure 5.3

From (Figure 5.3), and using equation (5.3.5), we can obtain the following system of ordinary differential equations:

$$\left.\begin{aligned} \frac{dS}{dt} &= -\beta SI + \alpha R \\ \frac{dE}{dt} &= \beta SI - \varepsilon E \\ \frac{dI}{dt} &= \varepsilon E - \gamma R \\ \frac{dR}{dt} &= \gamma R - \alpha R \end{aligned}\right\} \tag{5.4.1}$$

With initial conditions $S(0) = S_0$ , $E(0) = E_0$ , $I(0) = I_0$ , $R(0) = 0$

From adding the differential equations (5.4.1) we get:

$$\frac{dS}{dt} + \frac{dE}{dt} + \frac{dI}{dt} + \frac{dR}{dt} = 0$$

By taking integration, we get:

$$S + E + I + R = N \tag{5.4.2}$$

where $N$ is total population.

we have
$$R = N - S - E - I \tag{5.4.3}$$

Substituting equation (5.4.3) into the system (5.4.1) we get:

$$\left.\begin{aligned}\frac{dS}{dt} &= -\beta SI + \alpha(N - S - E - I)\\ \frac{dE}{dt} &= \beta SI - \varepsilon E\\ \frac{dI}{dt} &= \varepsilon E - \gamma(N - S - E - I)\end{aligned}\right\} \qquad (5.4.4)$$

By introducing the following new variables

$$S^* = \frac{S}{N}, \qquad E^* = \frac{E}{N}, \qquad I^* = \frac{\mathrm{I}}{N}, \qquad T = \varepsilon t$$

we have ( $S^*$ , $E^*$ , $I^*$ , $T$ ), the system (5.4.4) becomes the following system

$$\left.\begin{aligned}\frac{dS^*}{d\tau} &= -\lambda_1 S^* I^* + \lambda_2(1 - S^* - E^* - I^*)\\ \frac{dE^*}{d\tau} &= \lambda_1 S^* I^* - E^*\\ \frac{dI^*}{d\tau} &= E^* - \lambda_3(1 - S^* - E^* - I^*)\end{aligned}\right\} \qquad (5.4.5)$$

Where $\lambda_1 = \frac{\beta N}{\epsilon}$ , $\lambda_2 = \frac{\alpha}{\epsilon}$ , $\lambda_3 = \frac{\gamma}{\epsilon}$ .

## 6. CONCLUSION

Differential equations plays major role in applications of sciences and engineering, solved using either analytical or numerical methods. In this thesis, our main is to present shortly a differential equations and type of differential equations Let us formulate of nonlinear differential equations. Various types of stability may be discussed for the solutions of differential equations. The most important type is that concerning the stability of solutions near to a point of equilibrium. This may be discussed by the theory of Lyapunov. The SEIR Model is used in the modeling of infectious diseases by computing the amount of people in a closed population that are susceptible, exposed, infected, or recovered at a given period of time. The model is also used by researchers and health officials to explain the increase and decrease in people needing medical care for a certain disease during an epidemic.

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