REPUBLIC OF TURKEY

FIRAT UNIVERSITY

THE GRADUATE SCHOOL OF NATURAL AND APPLIED SCIENCE

APPLICATIONS OF THE SINE-GORDON EXPANSION METHOD TO THE SOME NONLINEAR PARTIAL DIFFERENTIAL EQUATIONS

AZAD PIRO SHAKIR
(142121101)

Master Thesis
Department: Mathematics
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Supervisor: Assoc. Prof. Dr. Hasan BULUT
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Supervisor : Assoc. Prof. Dr. Hasan BULUT(Firat Univ.)
Member: Assoc. Prof. Dr. Reşat YILMAZER (FiratUniv.)
Member: Assist.Prof. Dr. H. Mehmet BAŞKONUŞ (Munzur Univ.)

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SUMMARY

Applications of the sine-Gordon Expansion Method to the some nonlinear partial differential equations

This thesis consists six sections including the references.

In the first and second sections introduction and some fundamental definitions and theorems which are necessary in this study.

In third section, we present facts of the sine-Gordon Expansion Method.

In fourth, we apply the sine-Gordon Expansion Method to some know nonlinear solution equations, namely; (2+1)-dimensional Zakharov-Kuznetsov equation, Modified Benjamin-Bona-Mahony (mBBM) equation, Equal Width Wave equation, Modified Korteweg–de Vries (mKdV) equation. We obtain new solitary wave solution to the above mentioned models in form of hyperbolic function and trigonometric function solutions. All the computation in this study is carried out with Wolfram Mathematica 9.

We verify all the obtained solutions and they indeed all satisfied their corresponding equations, we also plot the two-dimensional and three-dimensional graphics, all with help of Wolfram Mathematica 9.

Finally, in the fifth section the detailed conclusion about the results obtained by sine-Gordon Expansion Method has been given.

ÖZET

Sine-Gordon Expantion metodunun bazı lineer olmayan kısmi differansiyel denklemlere uygulaması

Bu çalışma dört bölümden oluşmaktadır.

İlk bölümde, bu tezde kullanılan bazı temel tanım ve teoremler sunulmuştur.

İkinci bölümde, Sine-Gordon Expansion metodunun genel özellikleri açıklanmıştır.

Üçüncü bölümde, (2+1) boyutlu Zakharov-Kuznetsov denklemine, Benjamin–Bona–Mahony (BBM) denklemine, Equal width wave denklemine, modified Korteweg–de Vries (mKdV) denklemine Sine-Gordon Expansion metodu uygulanarak hiperbolik fonksiyon çözümleri, salınımlı dalga çözümleri elde edilmiştir. Ayrıca bu denklemlerin çözümlerinin iki ve üç boyutlu grafikler Wolfram Mathematica 9 programı kullanılarak çizilmiştir.

Dördüncü bölümde, Sine-Gordon Expansion metodu ile elde edilen çözümler hakkında ayrıntılı sonuç verilmiştir.

1. Introduction

Nonlinear equations have a great impress in engineering scopes and several scientific, fluid mechanics, chemical kinematics and geochemistry. Nonlinear wave phenomena of reaction are almost serious in nonlinear wave equations. That is important to look for the new solutions to for the nonlinear equations.

The (2+1)-dimensional Zakharov-Kuznetsov modified equal width equation is used to model the behavior of weakly nonlinear ion-acoustic waves in a plasma comprising cold ions and hot isothermal electrons in the presence of a uniform magnetic field [1,2]. Various approaches have been developed to investigate the solutions of (2+1)-dimensional Zakharov-Kuznetsov modified equal width equation.

In physical applications we see the Benjamin-Bona-Mahony (BBM) equation frequently because it expresses the model for propagation of long waves which embody nonlinear and dissipative effects; used to analysis the length of surface waves wavelength in liquids, hydro magnetic waves in cold plasma, acoustic-gravity waves in compressible fluids, and acoustic waves in harmonic crystals [3]. Many mathematicians paid their attention to the dynamics of the BBM equation [4].

Morrison and Meiss suggested to use the equal width (EW) wave equation as a model partial differential equation for the reproduction of one-dimensional wave breeding in a nonlinear medium with a distribution operation. [5]. Lot of numerical techniques advanced to solve the equation, such as the Adomain decomposition method, [6].

The existence of a shape-preserving solitary wave was established by Korteweg and de Vries and it takes shape as solution of an equation which bears the names. One of the equation that was first know to possess solitary wave type of solution is Korteweg-de Vries or KdV equation. There is still much activity going on the subject nowadays among the Physical and Mathematical communities [28].

2. FUNDAMENTAL DEFINITIONS AND THEOREMS

2.1 Definitions [17]

2.1.1 Definition of a partial differential equation (PDE)

The equation that contains the dependent variable and partial derivatives is said to be PDE. In the ordinary differential equations $u = u(x)$ the dependent variable depend on just one variable x which is independent. on other hand the partial Differential equation $u = u(x,t)$ or $u = u(x, y, t)$, should depend on further one independent variable.

If $u = u(x,t)$, the function $u = u(x,t)$ depends on x and t.

If $u = u(x, y, t)$, then the function $u = u(x, y, t)$ depends on x, y and t.

Some examples of partial differential equations:

$$\frac{\partial u}{\partial t} = k \frac{\partial^2 u}{\partial x^2}, \tag{2.1}$$

$$\frac{\partial u}{\partial t} = k \left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} \right), \tag{2.2}$$

$$\frac{\partial u}{\partial t} = k \left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2} \right), \tag{2.3}$$

that qualify in the above examples we have the heat flow equations in three types. The equation (2.1) is one dimensional and the dependent variable $u = u(x,t)$ depends on x and t. The equation (2.2) is two dimensional and the dependent variable $u = u(x, y, t)$ depends on x, y and t. The equation (2.3) is three dimensional and the dependent variable $u = u(x, y, z, t)$ depends on x, y, z and t.

Some different examples of partial differential equations:

$$\frac{\partial^2 u}{\partial t^2} = c^2 \frac{\partial^2 u}{\partial x^2}, \tag{2.4}$$

$$\frac{\partial^2 u}{\partial t^2} = c^2 \left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} \right), \tag{2.5}$$

$$\frac{\partial^2 u}{\partial t^2} = c^2 \left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2} \right), \tag{2.6}$$

that qualify in the above examples we have the wave propagation equations in three types. The Eq. (2.4) is the equation in one dimensional and the Eq. (2.5) is the equation in two dimensional and the Eq. (2.6) is the equation in three dimensional. These equations are defined by $u=u(x,t)$, $u=u(x,y,t)$, $u=u(x,y,z,t)$, depends on x,y,z and t.

We have the well-known Laplace equation in two dimensional and three dimensional:

$$\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}=0, \tag{2.7}$$

$$\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}+\frac{\partial^2 u}{\partial z^2}=0, \tag{2.8}$$

where the functions $u=u(x,y)$, $u=u(x,y,z)$, doesn't depend on t.

The Laplace's equation in polar coordinates:

$$\frac{\partial^2 u}{\partial r^2}+\frac{\partial u}{r.\partial r}+\frac{\partial^2 u}{r^2.\partial \theta^2}=0, \tag{2.9}$$

where $u=u(r,\theta)$.

The Korteweg–de Vries equation:

$$\frac{\partial u}{\partial t}+6u\frac{\partial u}{\partial x}+\frac{\partial^3 u}{\partial x^3}=0, \tag{2.10}$$

where the function $u=u(x,t)$, depends on variable x and time variable t.

The Burgers equation:

$$\frac{\partial u}{\partial t}+u\frac{\partial u}{\partial x}-v\frac{\partial^2 u}{\partial x^2}=0, \tag{2.11}$$

where the function $u=u(x,t)$, depends on variable x and time variable t.

2.1.2 Order of a PDE

We called the order of the highest derivative that we get in the equation the order of equation.

The example for PDE of first order:

$$\frac{\partial u}{\partial x}-\frac{\partial u}{\partial y}=0. \tag{2.12}$$

The example for PDE of second order:

$$\frac{\partial^2 u}{\partial x^2} - \frac{\partial u}{\partial t} = 0. \tag{2.13}$$

The example for PDE of third order:

$$\frac{\partial u}{\partial y} - u\frac{\partial^3 u}{\partial x^3} = 0. \tag{2.14}$$

2.1.3 Linear and Nonlinear PDEs

Partial differential equations have to type linear and nonlinear. If a partial differential equation satisfies two following condition is said to be linear:

First condition: The power of each dependent variable and partial derivative that obtained in equation should be one.

Second condition: the coefficients must be constant or independent for the dependent variable also any partial derivative.

The equation is not linear if which of the condition not satisfied.

Examples for linear equation is:

$$(1)\ x\frac{\partial^2 u}{\partial x^2} + y\frac{\partial^2 u}{\partial y^2} = 0.$$

$$(2)\ \frac{\partial^2 u}{\partial r^2} + \frac{1}{r}\frac{\partial u}{\partial r} + \frac{1}{r^2}\frac{\partial^2 u}{\partial \theta^2} = 0.$$

Examples for nonlinear equation is:

$$(1)\ u\frac{\partial u}{\partial t} + x\frac{\partial u}{\partial x} = 2.$$

$$(2)\ \frac{\partial u}{\partial x} + \sqrt{u} = x.$$

2.1.4 Some Linear PDEs

As mention before Linear PDEs used many filed of scientific applications, like wave and diffusion equation. The following are the famous models:

1. The wave equation:

$$\frac{\partial^2 u}{\partial t^2} = c^2\frac{\partial^2 u}{\partial x^2}, \qquad \text{where } c \text{ is arbitrary constant.}$$

2.The heat equation:

$$\frac{\partial u}{\partial t} = k\frac{\partial^2 u}{\partial x^2}, \qquad \text{where } k \text{ is arbitrary constant.}$$

3. The Klein-Gordon equation:

$$\nabla^2 u - \frac{\partial^2 u}{c^2.\partial t^2} = \mu^2 u, \quad \text{where } \mu \text{ and } c \text{ are arbitrary constants.}$$

4. The Laplace equation:

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0,$$

5. The Telegraph equation:

$$\frac{\partial^2 u}{\partial x^2} = a\frac{\partial^2 u}{\partial t^2} + b\frac{\partial^2 u}{\partial t^2} + cu, \qquad \text{where } a, b \text{ and } c \text{ are arbitrary constants.}$$

6. The Linear Schrodinger's equation:

$$i\frac{\partial u}{\partial t} + \frac{\partial^2 u}{\partial x^2} = 0, \quad i = \sqrt{-1}.$$

2.1.5 Some Nonlinear PDEs

The nonlinear used in physic, mathematics and engineering like fluid dynamics and quantum field theory. The following are the famous models:

1. The Burgers equation:

$$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} = \alpha\frac{\partial^2 u}{\partial x^2}.$$

2. The Advection equation:

$$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} = f(x,t).$$

3. The Korteweg de-Vries equation:

$$\frac{\partial u}{\partial t} + \alpha u\frac{\partial u}{\partial x} + b\frac{\partial^3 u}{\partial x^3} = 0.$$

4. The modified Korteweg de-Vries equation:

$$\frac{\partial u}{\partial t} - 6u^2\frac{\partial u}{\partial x} + \frac{\partial^3 u}{\partial x^3} = 0.$$

5. The Sine-Gordon equation:

$$\frac{\partial^2 u}{\partial t^2} - \frac{\partial^2 u}{\partial x^2} = \alpha \sin(u).$$

6. The sinh-Gordon equation:

$$\frac{\partial^2 u}{\partial t^2}-\frac{\partial^2 u}{\partial x^2}=\alpha \sinh(u).$$

7. The Boussinesq equation:

$$\frac{\partial^2 u}{\partial t^2}-\frac{\partial^2 u}{\partial x^2}+3\frac{\partial^2 u^2}{\partial x^2}-\frac{\partial^3 u}{\partial x^3}=0.$$

8. The Fisher equation:

$$\frac{\partial u}{\partial t}=D\frac{\partial^2 u}{\partial x^2}+u(1-u).$$

9. The Liouville equation:

$$\frac{\partial^2 u}{\partial t^2}-\frac{\partial^2 u}{\partial x^2}=e^{\pm u}.$$

10. The K (n, n) equation:

$$\frac{\partial u}{\partial t}+\alpha\frac{\partial u^n}{\partial x}+b\frac{\partial u^n}{\partial x^2}=0,\quad n>1.$$

11. The Nonlinear Schrodinger equation:

$$i\frac{\partial u}{\partial t}+\frac{\partial^2 u}{\partial x^2}+\gamma|u|^2 u=0.$$

12. The Camassa-Holm equation:

$$\frac{\partial u}{\partial t}-\frac{\partial^3 u}{\partial x^2\partial t}+\alpha\frac{\partial u}{\partial x}+3u\frac{\partial u}{\partial x}=2\frac{\partial u}{\partial x}\frac{\partial^2 u}{\partial x^2}+u\frac{\partial^3 u}{\partial x^3}.$$

13. The Degasperis-Procesi equation:

$$\frac{\partial u}{\partial t}-\frac{\partial^3 u}{\partial x^2\partial t}+\alpha\frac{\partial u}{\partial x}+4u\frac{\partial u}{\partial x}=3\frac{\partial u}{\partial x}\frac{\partial^2 u}{\partial x^2}+u\frac{\partial^3 u}{\partial x^3}.$$

14. The Kadomtsev-Petviashvili equation:

$$\frac{\partial}{\partial x}\left(\frac{\partial u}{\partial t}+\alpha u\frac{\partial u}{\partial x}+b\frac{\partial^3 u}{\partial x^3}\right)+\frac{\partial^2 u}{\partial y^2}=0.$$

2.1.6 Homogeneous and Inhomogeneous PDEs

There are two types of partial differential equations homogeneous or inhomogeneous. It said to be homogeneous if it has variable u which is dependent or one of it is derivatives, However, if any of these conditions is not satisfied, it is said to be an inhomogeneous partial differential equation.

Example for PDEs are homogeneous.

$$(1)\ \frac{\partial u}{\partial t} = 4\frac{\partial^2 u}{\partial x^2}$$

$$(2)\ \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$$

Example for PDEs are an inhomogeneous.

$$(1)\ \frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2} + x$$

$$(2)\ \frac{\partial u}{\partial x} + \frac{\partial u}{\partial y} = u + 4$$

2.1.7 Solution of a Partial differential equation

We called u is a partial differential equation solution if we put the result of solution instead of u right hand side and left hand side of equation should be equal.

Remarks:

In the following we will show the properties of a partial differential equation solution.

1. If ordinary differential equation is linear homogeneous it obvious if $u_1, u_2, ..., u_n$ are the solutions of the equation, then a linear combination of $u_1, u_2, ...$ that has the form:

$$u = c_1u_1 + c_2u_2 + ... + c_nu_n, \qquad (2.15)$$

is also a solution. The term superposition principle named for combining two or more of this solution.

We can note that the superposition principle for PDE when it's homogeneous works effectively in a given domain.

2. In linear partial differential equation, general solution depends on spot function. On other hand the solution of ordinary differential equations depends on spot constant. It will be easy to see this in the following if the partial differential equation.

$$\frac{\partial u}{\partial x} + \frac{\partial u}{\partial y} = 0. \qquad (2.16)$$

When it's solution is:

$$u = f(x - y), \qquad (2.17)$$

where $f(x-y)$ is an arbitrary differentiable function. So the solution of equation (2.16) will be following:

$$
\begin{aligned}
&u = x - 1, \\
&u = e^{x-y}, \\
&u = \sinh(x-y), \\
&u = \ln(x-y).
\end{aligned} \tag{2.18}
$$

Actually particular solution constantly desired to satisfied prescribed conditions.

2.2 Second-order PDE

The general form of partial differential equation for second order linear in two variables x , y is:

$$A\frac{\partial^2 u}{\partial x^2} + B\frac{\partial^2 u}{\partial x \partial y} + C\frac{\partial^2 u}{\partial y^2} + D\frac{\partial u}{\partial x} + E\frac{\partial u}{\partial y} + Fu = G, \tag{2.19}$$

where A, B, C, D, E, F and G are arbitrary constants.

We can classified the PDE of second order (2.19) into the following:

1. Parabolic: If $B^2 - 4AC = 0,$ then the equation is said to be Parabolic equation.

The heat transfer equation is example for parabolic equations.

$$\frac{\partial u}{\partial t} = K\frac{\partial^2 u}{\partial x^2}.$$

2. Hyperbolic: If $B^2 - 4AC > 0,$ then the equation is said to be Hyperbolic equation.

The wave equation is example for Hyperbolic equations.

$$\frac{\partial^2 u}{\partial t^2} = c^2\frac{\partial^2 u}{\partial x^2}.$$

3. Elliptic: If $B^2 - 4AC < 0,$ then the equation is said to be Elliptic equation.

The Laplace equation in a two dimensional is example for Elliptic equations.

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0.$$

2.2 THEOREMS [27]

Problem (1). Prove that on the interval $-1 \le t \le 1$, the initial-value following problem has a solution.

$$\begin{cases} x' = (t + \sin x)^2 \\ x(0) = 3. \end{cases}$$

Solution: In this example, $f(t,x) = (t+\sin x)^2$ and $(t_0, x_0) = (0,3)$. In the rectangle

$$R = \{(t,x) : |t| \le \alpha, |x-3| \le \beta\},$$

the magnitude of f is bounded by:

$$|f(t,x)| \le (\alpha+1)^2 = M.$$

We want $\min(\alpha, \beta/M) \ge 1$, and so we can let $\alpha = 1$. Then $M = 4$, and our objective is met by letting $\beta \ge 4$. The existence theorem asserts that on the interval $|t| \le \min(\alpha, \beta/M) = 1$ a solution of the initial value problem exists.

THEOREM (a). If f is continuous in a rectangle R centered at (t_0, x_0), say

$$R = \{(t,x) : |t-t_0| \le \alpha, \ |x-x_0| \le \beta\}. \tag{2.20}$$

Then the initial-value problem (1) has a solution $x(t)$ for $|t-t_0| \le \min\left(\alpha, \ \dfrac{\beta}{M}\right)$, where M is the $\max(|f(t,x)|)$ in the rectangle R.

THEOREM (b). If function f and $\dfrac{\partial f}{\partial x}$ are continuous in the rectangle $R = \{(t,x) : |t-t_0| \le \alpha, \ |x-x_0| \le \beta\}$, in the interval $|t-t_0| \le \min\left(\alpha, \ \dfrac{\beta}{M}\right)$, then the initial-value problem (1) has a unique solution.

In the both of Theorem (a) and (b), the interval on the t-axis in which the solution is asserted to exist may be smaller than the base of the rectangle in which we have defined $f(t,x)$. Theorem (c) is of a different type that allows us to infer the existence and uniqueness of a solution on a prescribed interval $[a,b]$. (See Henrici [1962, p. 15])

THEOREM (c). In the strip $a \le t \le b, -\infty < x < \infty$**,** if function f is continuous and the function satisfies there an inequality:

$$\left| f(t, x_1) - f(t, x_2) \right| \le L \left| x_1 - x_2 \right|. \tag{2.21}$$

In the interval $[a, b]$, the initial-value problem (1) has a unique solution.

The inequality equation (2.21) is said to be Lipschitz condition in the second variable. For a function of one variable, such a condition would assert simply

$$\left| g(x_1) - g(x_2) \right| \le L \left| x_1 - x_2 \right|. \tag{2.22}$$

We see promptly that this condition is more grounded than continuity, for if x_2 approaches x_1, the right-hand side Ex. (2.22) approaches zero, and this forces $g(x_2)$ to approaches $g(x_1)$. The condition Ex. (2.22) is weaker than having a bounded derivative. Indeed, if $g'(x)$ exist everywhere and does not exceed L in modulus, then by Mean-value Theorem

$$\left| g(x_1) - g(x_2) \right| = \left| g'(\xi) \right| \left| x_1 - x_2 \right| \le L \left| x_1 - x_2 \right|.$$

3. THE SINE-GORDON EXPANSION METHOD (SGEM)

Let us consider the following sine-Gordon equation [18,19,20,23];

$$u_{xx} - u_{tt} = m^2 \sin(u). \tag{3.1}$$

When m is real arbitrary constant and u equal to $u(x,t)$. When the applying wave transform $\xi = \mu(x - ct)$ to Eq. (3.1), we will get the ordinary differential equation which is nonlinear follows:

$$U'' = \frac{m^2}{\mu^2(1-c^2)} \sin(U), \tag{3.2}$$

where $U = U(\xi)$, ξ is the amplitude of the travelling wave and c is the velocity of the travelling wave. After we rewrite Eq. (3.2), we obtain follows equation:

$$\left[\left(\frac{U}{2}\right)'\right]^2 = \frac{m^2}{\mu^2(1-c^2)} \sin^2\left(\frac{U}{2}\right) + K, \tag{3.3}$$

where K is the integration constant. If we put values of $K = 0$, $w(\xi) = \frac{U}{2}$, and $a^2 = \frac{m^2}{\mu^2(1-c^2)}$ in Eq. (3.3), the equation will be as follow:

$$w' = a \sin(w). \tag{3.4}$$

We locate value $a = 1$ in Eq. (3.4), the equation will be as follow:

$$w' = \sin(w). \tag{3.5}$$

After that, we solve the Eq. (3.5) by using the method (separation of variables), we obtain this two significant equations:

$$\sin(w) = \sin\left[w(\xi)\right] = \left.\frac{2pe^{\xi}}{p^2 e^{2\xi} + 1}\right|_{p=1} = \sec h(\xi), \tag{3.6}$$

$$\cos(w)=\cos(w(\xi))=\left.\frac{p^2e^{2\xi}-1}{p^2e^{2\xi}+1}\right|_{p=1}=\tanh(\xi), \tag{3.7}$$

where $p\neq 0,$ is the integral constant. For the find solution of this nonlinear partial differential equation:

$$P(u,u_x,u_t,\cdots)=0. \tag{3.8}$$

Let us consider:

$$U(\xi)=\sum_{i=1}^{n}\tanh^{i-1}(\xi)\left[B_i\sec h(\xi)+A_i\tanh(\xi)\right]+A_0. \tag{3.9}$$

We write the Eq. (3.9) similar to the Eqs. (3.6) and (3.7) as:

$$U(w)=\sum_{i=1}^{n}\cos^{i-1}(w)\left[B_i\sin(w)+A_i\cos(w)\right]+A_0. \tag{3.10}$$

By using the balance technique, we can find the values of n. Allow the coefficients of the equation $\sin^i(w)\cos^j(w)$ be zero. Solving this system with programming (Wolfram Mathematica) and give the values of A_i,B_i,c,μ. At last, we can substituting the values of A_i,B_i,c,μ in Eq. (3.9), we obtain the new analytical solutions for the Eq. (3.8).

4. APPLICATIONS OF THE METHOD

4.1. Application of SGEM to the (2+1)-dimensional Zakharov-Kuznetsov modified equal width equation [1,2]

Let's consider the (2+1)-dimensional Zakharov-Kuznetsov modified equal width equation defined as following;

$$u_t + 3au^2u_x + b\left(u_{xxt} + u_{xyy}\right) = 0, \tag{4.1.1}$$

where a, b are constant and non-zero. If we apply the following transformation,

$$u = U(\xi),\ \xi = \mu(x + y - 2kt),$$

we can find the following nonlinear ordinary differential equation for equation (4.1.1)

$$b\mu^2(1-2k)U''' + 3aU^2U' - 2kU' = 0, \tag{4.1.2}$$

where $u_t = -2\mu kU', u_x = \mu U', u_{xxt} = -2\mu^3 kU''', u_{xxy} = \mu^3 U'''.$ Taking integration of equation (4.1.2), we obtain the following equation;

$$b\mu^3(1-2k)U'' + a\mu U^3 - 2k\mu U = 0. \tag{4.1.3}$$

After we divide the equation (4.1.3) by μ, we will get the nonlinear ordinary differential equation as follow for equation (4.1.1).

$$b\mu^2(1-2k)U'' + aU^3 - 2kU = 0. \tag{4.1.4}$$

If we apply balance principle for equation (4.1.4) between U'' and U^3, we get the following values n.

$$n = 1. \tag{4.1.5}$$

When we put equation (4.1.5) in equation (3.10), we obtain following travelling wave solution for equation (4.1.4) as

$$U(w) = B_1\sin(w) + A_1\cos(w) + A_0. \tag{4.1.6}$$

If we consider equation (4.1.6) for second derivation of (4.1.4), we can write follows;

$$U''(w)=B_1\cos^2(w)\sin(w)-B_1\sin^3(w)-2A_1\sin^2(w)\cos(w). \tag{4.1.7}$$

If we consider equations (4.1.6), (4.1.7) for equation (4.1.4), we get the following equation;

$$\begin{aligned}&b\mu^2(1-2k)\left[B_1\cos^2(w)\sin(w)-B_1\sin^3(w)-2A_1\sin^2(w)\cos(w)\right]\\&+a\left[B_1\sin(w)+A_1\cos(w)+A_0\right]^3-2k\left[B_1\sin(w)+A_1\cos(w)+A_0\right]=0.\end{aligned} \tag{4.1.8}$$

We can rewrite equation (4.1.8) as following;

$$\begin{aligned}&aA_0^3-2kA_0-2kA_1\cos(w)-2b\mu^2A_1\sin^2(w)\cos(w)\\&+4kb\mu^2A_1\sin^2(w)\cos(w)+3aA_0^2A_1\cos(w)+3aA_1^2A_0\cos^2(w)\\&+aA_1^3\cos^3(w)-2kB_1\sin(w)+b\mu^2B_1\cos^2(w)\sin(w)\\&-2kb\mu^2B_1\cos^2(w)\sin(w)+2kb\mu^2B_1\sin^3(w)-b\mu^2B_1\sin^3(w)\\&+3aA_0^2B_1\sin(w)+6aB_1A_1A_0\sin(w)\cos(w)+3aA_1^2B_1\cos^2(w)\sin(w)\\&+3aB_1^2A_0\sin^2(w)+3aB_1^2A_1\sin^2(w)\cos(w)+aB_1^3\sin^3(w)=0.\end{aligned} \tag{4.1.9}$$

After then, when we sort out equation (4.1.9) according to trigonometric functions, we can type follows;

$$\begin{aligned}&constant &&: aA_0^3-2kA_0+3aA_1^2A_0=0,\\&\sin(w) &&: -2kB_1+3aA_0^2B_1-b\mu^2B_1+2kb\mu^2B_1+aB_1^3=0,\\&\cos(w) &&: -2kA_1+3aA_0^2A_1+aA_1^3=0,\\&\sin(w)\cos(w) &&: 6aB_1A_1A_0=0,\\&\sin^2(w) &&: 3aB_1^2A_0-3aA_1^2A_0=0,\\&\sin^2(w)\cos(w) &&: -2b\mu^2A_1+4kb\mu^2A_1+3aB_1^2A_1-aA_1^3=0,\\&\sin^3(w) &&: -4kb\mu^2B_1+2b\mu^2B_1-aB_1^3+3aA_1^2B_1=0.\end{aligned} \tag{4.1.10}$$

If we solve this system by using the programming (Wolfram Mathematica), we can find the following coefficients which give us many different travelling wave solutions for equation (4.1.1);

Case-1

$$A_0 = 0, A_1 = \frac{\sqrt{b}\mu}{\sqrt{a\left(b\mu^2 - 2\right)}}, B_1 = -\frac{\sqrt{b}\mu}{\sqrt{a\left(2 - b\mu^2\right)}}, k = \frac{1}{2} + \frac{1}{\left(b\mu^2 - 2\right)}. \tag{4.1.11}$$

If we get the coefficients of equation (4.1.11) for equation (3.9), we obtain:

$$u_1(x,y,t) = \sqrt{b}\mu\left(-\frac{\operatorname{sech}\left[\mu\left(x+y-2t\left(\frac{1}{2}+\frac{1}{\left(b\mu^2-2\right)}\right)\right)\right]}{\sqrt{a\left(2-b\mu^2\right)}} - \frac{\tanh\left[\mu\left(x+y-2t\left(\frac{1}{2}+\frac{1}{\left(b\mu^2-2\right)}\right)\right)\right]}{i\sqrt{a\left(2-b\mu^2\right)}}\right) \tag{4.1.12}$$

where a, b, μ are real constants and not zero.

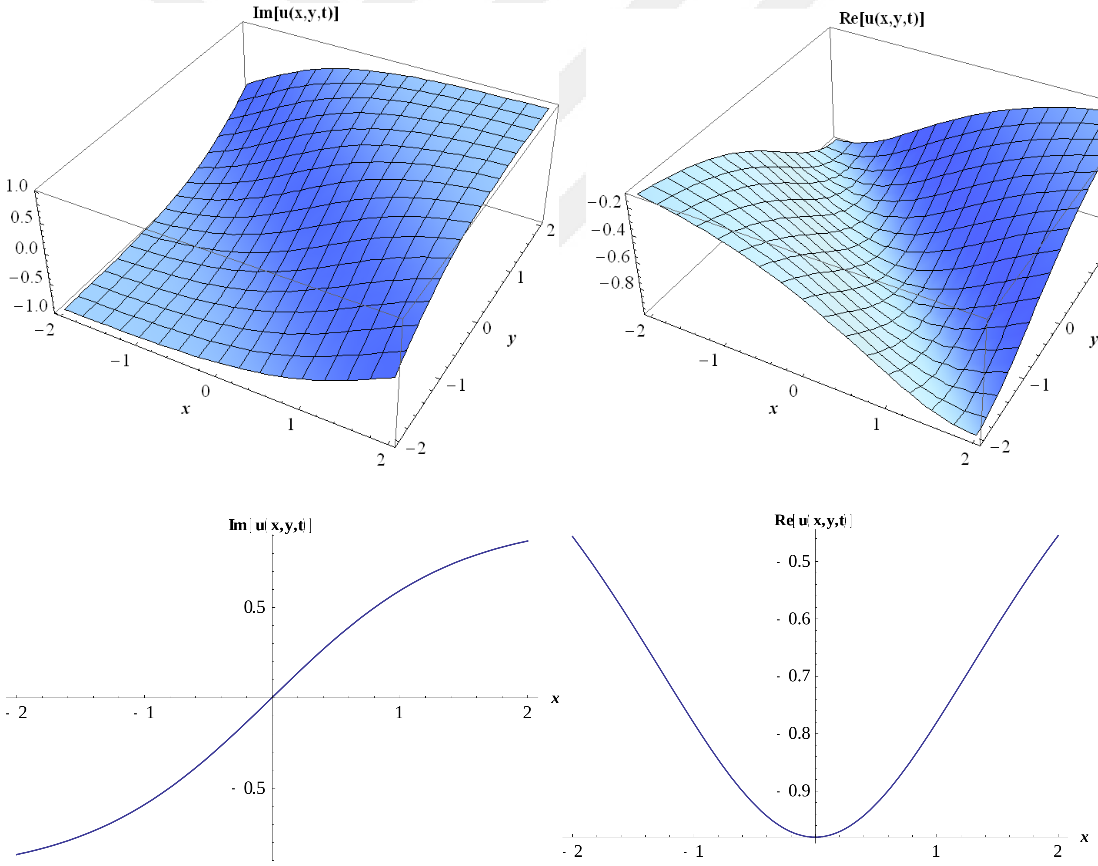


Figure.1 The three dimensional and two dimensional surfaces for Eq.(4.1.12) by the values $a = 1, b = 2,\ \mu = 0.7, t = 0.002, -2 < x < 2,\ -2 < y < 2$ and $y = 0.001$ for 2D graphics.

Case-2

$$A_0 = 0, A_1 = 0, B_1 = -\frac{\sqrt{2}\sqrt{b}\mu}{\sqrt{a\left(1+b\mu^2\right)}}, k = \frac{b\mu^2}{2\left(1+b\mu^2\right)}. \tag{4.1.13}$$

If we get the coefficients of equation (4.1.13) for equation (4.9), we obtain:

$$u_2\left(x,y,t\right) = -\frac{\sqrt{2}\sqrt{b}\mu \operatorname{sech}\left[\mu\left(x+y-\frac{2bt\mu^2}{2\left(1+b\mu^2\right)}\right)\right]}{\sqrt{a\left(1+b\mu^2\right)}}. \tag{4.1.14}$$

Where a, b, μ are real constants and non-zero.

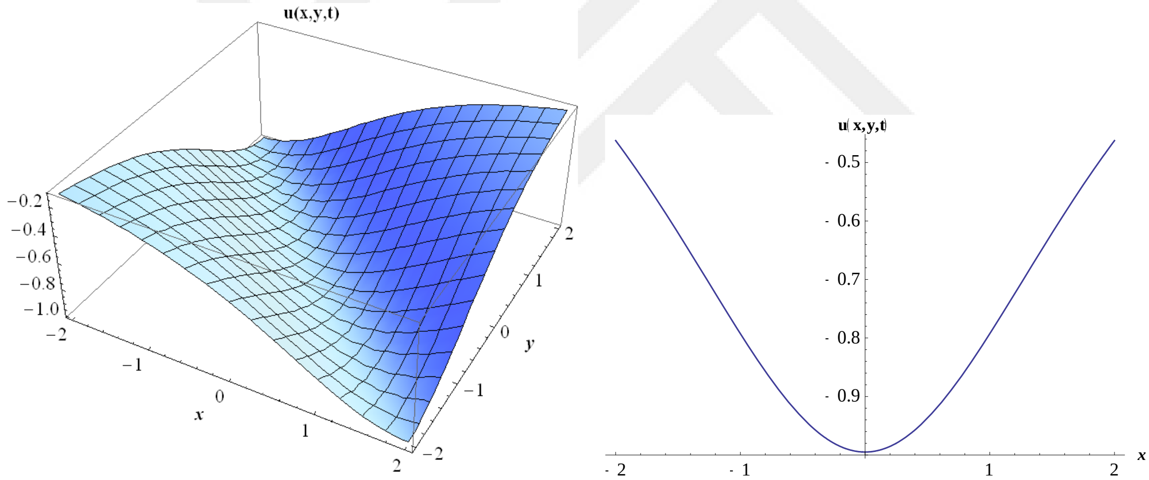


Figure.2 The three dimensional and two dimensional surfaces for Eq.(4.1.14) by the values $a = 1,\ b = 2, \mu = 0.7, t = 0.002, -2 < x < 2,\ -2 < y < 2$ and $y = 0.001$ for 2D graphics.

Case-3

$$A_0 = 0, A_1 = 0, B_1 = \frac{2\sqrt{k}}{\sqrt{a}}, b = \frac{2k}{\mu^2\left(2k-1\right)}. \tag{4.1.15}$$

If we get the coefficients of equation (4.1.15) for equation (3.9), we obtain:

$$u_3(x,y,t) = -\frac{2\sqrt{k}\,\mathrm{sech}\left[\mu\left(x+y-2kt\right)\right]}{\sqrt{a}}. \tag{4.1.16}$$

Where a, b, μ are real constants and non-zero.

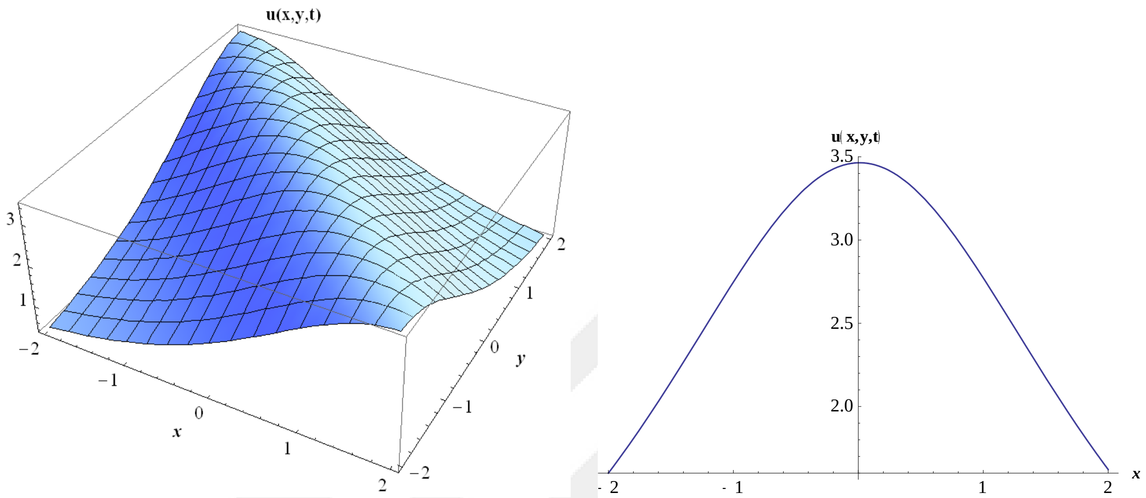


Figure.3 The three dimensional and two dimensional surfaces for Eq.(4.1.16) by the values $a=1,\ k=3, \mu=0.7, t=0.002, -2<x<2,\ -2<y<2$ and $y=0.001$ for 2D graphics.

Case-4

$$A_0 = 0, A_1 = -\frac{\sqrt{2}\sqrt{k}}{a}, B_1 = -\frac{i\sqrt{2}\sqrt{k}}{\sqrt{a}}, b = \frac{4k}{\mu^2\left(2k-1\right)}. \tag{4.1.17}$$

If we get the coefficients of equation (4.1.17) for equation (3.9), we obtain:

$$u_4(x,y,t) = \frac{\sqrt{2}\sqrt{k}}{\sqrt{a}}\left(-i\,\mathrm{sech}\left[\mu\left(x+y-2kt\right)\right] - \tanh\left[\mu\left(x+y-2kt\right)\right]\right). \tag{4.1.18}$$

Where a, b, μ is real constants and non-zero.

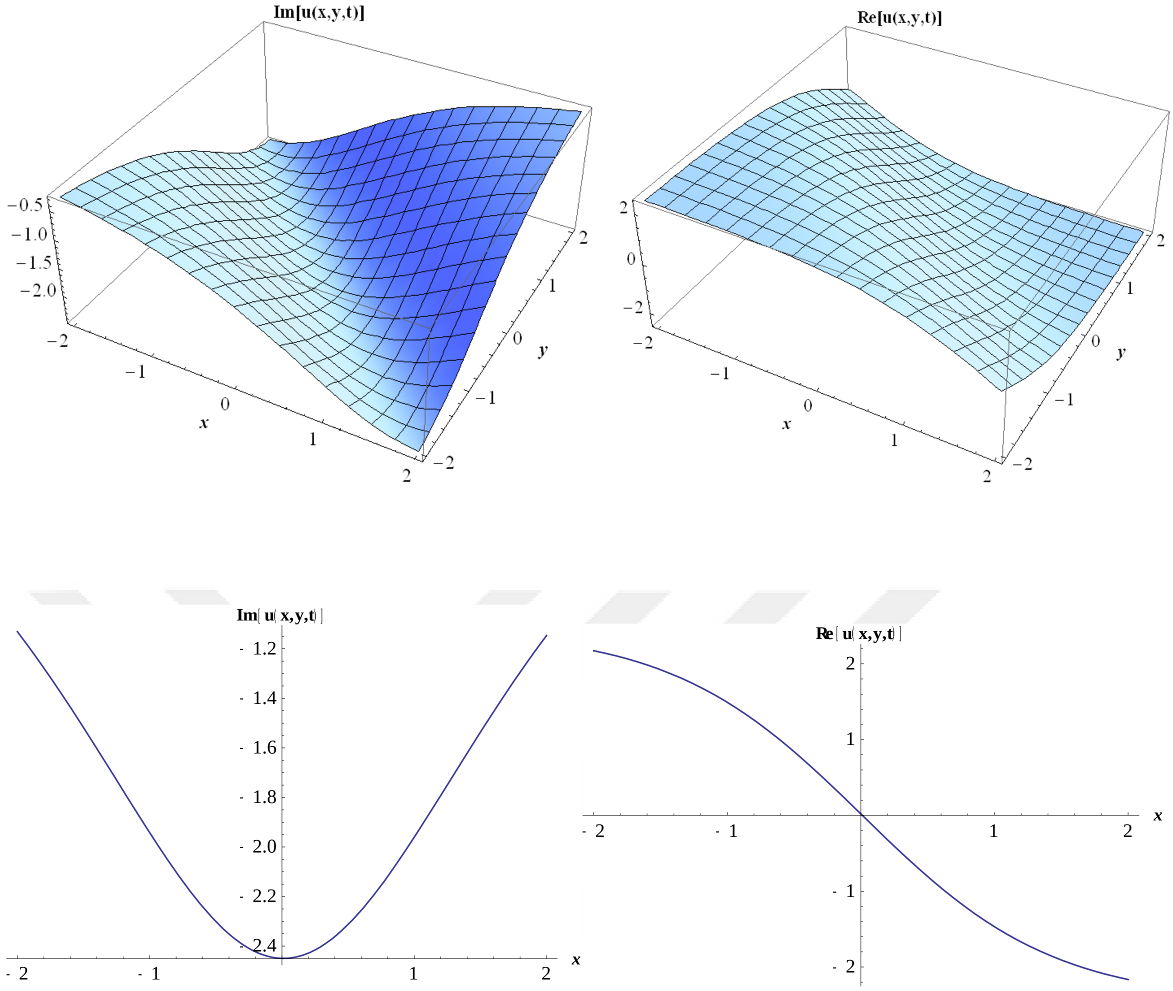


Figure.4 The three dimensional and two dimensional surfaces for Eq.(4.1.18) by the values $a=1,\ k=3, \mu=0.7, t=0.002, -2<x<2,\ -2<y<2$ and $y=0.001$ for 2D graphics.

4.2. Application of SGEM to the modified Benjamin–Bona–Mahony (mBBM) equation

Let us consider the mBBM equation defined as following [21];

$$u_t + u_x + au^2 u_x + bu_{xxt} = 0, \tag{4.2.1}$$

where a, b are constant and non-zero. If we apply the following transformation,

$$u = u(x,t) = U(\xi),\ \xi = \mu(x - ct), \tag{4.2.2}$$

we can get the following nonlinear ordinary differential equation for equation (4.2.1)

$$-c\mu U' + \mu U' + aU^2 \mu U' - bc\mu^3 U''' = 0, \tag{4.2.3}$$

where $u_t = -c\mu U', u_x = \mu U', u_{xxt} = -c\mu^3 U'''$. Taking integration of equation (4.2.3), we get the following equation;

$$-c\mu U + \mu U + a\mu \frac{U^3}{3} - bc\mu^3 U'' = 0. \tag{4.2.4}$$

Multiply 3 and divide by μ to the equation (4.2.4), we obtain the following nonlinear ordinary differential equation for equation (4.2.1).

$$(3 - 3c)U + aU^3 - 3bc\mu^2 U'' = 0. \tag{4.2.5}$$

If we apply balance principle for equation (4.2.5) between U'' and U^3, we can get the following values n.

$$n = 1. \tag{4.2.6}$$

When we put equation (4.2.6) in equation (3.10), we obtain following travelling wave solution for equation (4.2.4) as

$$U(w) = B_1 \sin(w) + A_1 \cos(w) + A_0. \tag{4.2.7}$$

If we consider equation (4.2.7) for second derivation of (4.2.5), we can write follows;

$$U''(w) = B_1 \cos^2(w)\sin(w) - B_1 \sin^3(w) - 2A_1 \sin^2(w)\cos(w). \tag{3.2.8}$$

If we consider equations (4.2.7), (4.2.8) for equation (4.2.5), we can find the following equation;

$$\begin{aligned}&-3c\left(B_1\sin(w)+A_1\cos(w)+A_0\right)+3\left(B_1\sin(w)+A_1\cos(w)+A_0\right)\\&+a\left(B_1\sin(w)+A_1\cos(w)+A_0\right)^3-3bc\mu^2[B_1\cos^2\left(w\right)\sin\left(w\right)\\&-B_1\sin^3\left(w\right)-2A_1\sin^2\left(w\right)\cos\left(w\right)]=0.\end{aligned} \tag{4.2.9}$$

We can rewrite equation (4.2.9) as following;

$$\begin{aligned}&-3cB_1\sin(w)-3cA_1\cos(w)+3B_1\sin(w)+3A_1\cos(w)\\&+aB_1^3\sin^3(w)+aA_1^3\cos^3\left(w\right)+6aB_1A_1A_0\sin\left(w\right)\cos\left(w\right)\\&+3aB_1^2A_1\sin^2\left(w\right)\cos\left(w\right)+3aA_1^2B_1\cos^2\left(w\right)\sin(w)+3A_0\\&+3aB_1^2A_0\sin^2(w)+3aA_1^2A_0\cos^2(w)+3aA_0^2B_1\sin(w)\\&+3aA_0^2A_1\cos(w)-3bc\mu^2B_1\cos^2\left(w\right)\sin\left(w\right)-3cA_0+aA_0^3\\&+3bc\mu^2B_1\sin^3\left(w\right)+6bc\mu^2A_1\sin^2\left(w\right)\cos\left(w\right)=0.\end{aligned} \tag{4.2.10}$$

After then, when we sort out equation (4.2.10) according to trigonometric functions, we can type follows;

$$\begin{aligned}&constant &&: -3cA_0+3A_0+aA_0^3+3aA_1^2A_0=0,\\&\sin(w) &&: -3cB_1+3B_1+3aA_0^2B_1+aB_1^3+3bc\mu^2B_1=0,\\&\cos\left(w\right) &&: -3cA_1+3A_1+3aA_0^2A_1+aA_1^3=0,\\&\sin(w)\cos(w) &&: 6aB_1A_1A_0=0,\\&\sin^2(w) &&: 3aB_1^2A_0-3aA_1^2A_0=0,\\&\sin^2\left(w\right)\cos\left(w\right) &&: 3aB_1^2A_1+6bc\mu^2A_1-aA_1^3=0,\\&\cos^2\left(w\right)\sin\left(w\right) &&: 3aA_1^2B_1-6bc\mu^2B_1-aB_1^3=0.\end{aligned} \tag{4.2.11}$$

If we solve this system by using the programming (Wolfram Mathematica), we can find the following coefficients which give us many different travelling wave solutions for equation (4.2.1);

Case-1

$$A_0 = 0, A_1 = \frac{-1}{\sqrt{a}}\sqrt{3(-1+c)}, B_1 = 0, b = \frac{-1+c}{2c\mu^2}. \tag{4.2.12}$$

If we get the coefficients of equation (4.2.12) for equation (3.9), we obtain the hyperbolic travelling wave solution for the mBBM equation as following;

$$u_1(x,t) = \frac{-1}{\sqrt{a}}\sqrt{3(-1+c)}\tanh(\mu x - c\mu t). \tag{4.2.13}$$

where a, c, μ are real constants and not zero.

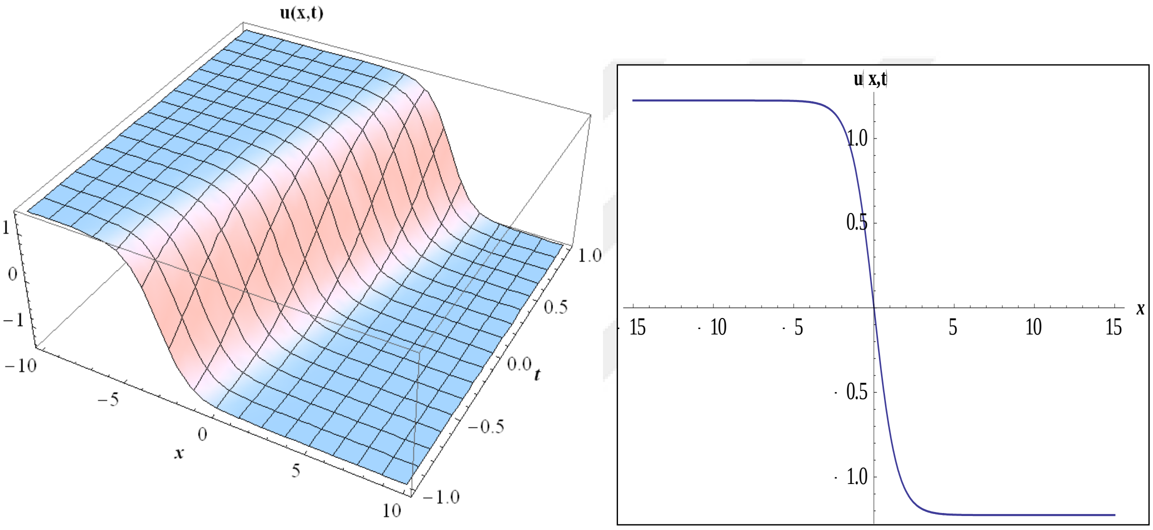


Figure.5 The three dimensional and two dimensional surfaces for Eq.(4.2.13) by considering the values $a = 3, c = 3,\ \mu = 0.7, -10 < x < 10,\ -1 < t < 1$, and for two dimensional $t = 0.002$.

Case-2

$$A_0 = 0, A_1 = \frac{1}{\sqrt{a}}\sqrt{3(-1+c)}, B_1 = 0, \mu = \sqrt{\frac{-1+c}{2cb}}. \tag{4.2.14}$$

If we get the coefficients of equation (4.2.14) for equation (3.9), we obtain the hyperbolic travelling wave solution for the mBBM equation as following;

$$u_2(x,t) = \frac{1}{\sqrt{a}}\sqrt{3(-1+c)}\tanh\left(\sqrt{\frac{-1+c}{2cb}}x - c\sqrt{\frac{-1+c}{2cb}}t\right). \tag{4.2.15}$$

Where a, c, b are real constants and not zero.

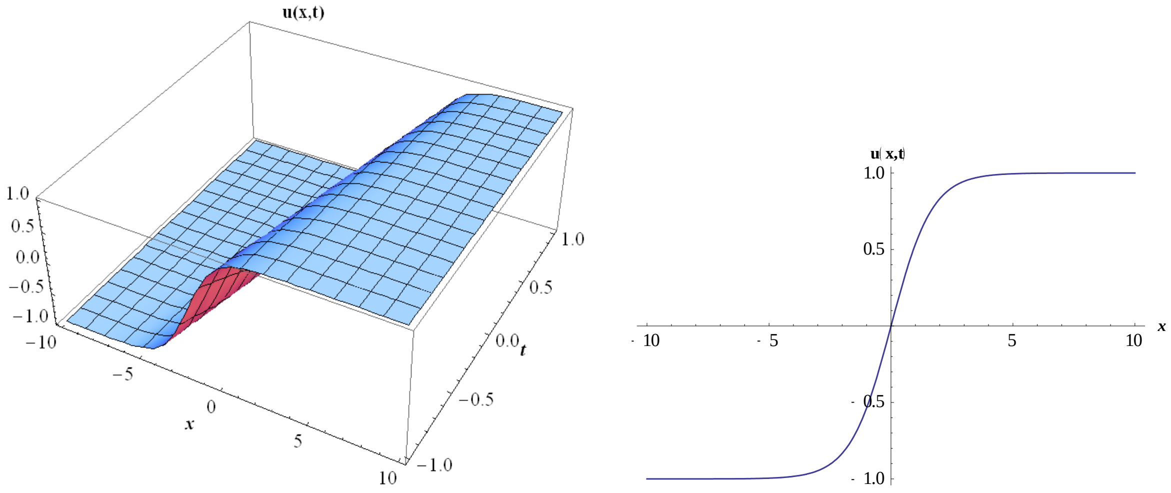


Figure.6 The three dimensional and two dimensional surfaces for Eq.(4.2.15) by considering the values $c=2$, $b=0.7, a=3, -10<x<10,\ \ -1<t<1$ and for two dimensional $t=0.002$.

Case-3

$$A_0=0, A_1=0, B_1=\frac{1}{\sqrt{a}}\sqrt{6(-1+c)}, b=\frac{1-c}{c\mu^2}. \tag{4.2.16}$$

If we get the coefficients of equation (4.2.16) for equation (3.9), we obtain the hyperbolic travelling wave solution for the mBBM equation as following;

$$u_3(x,t)=\frac{1}{\sqrt{a}}\sqrt{6(-1+c)}\operatorname{sech}(\mu x-ct\mu). \tag{4.2.17}$$

Where a,c,μ are real constants and not zero.

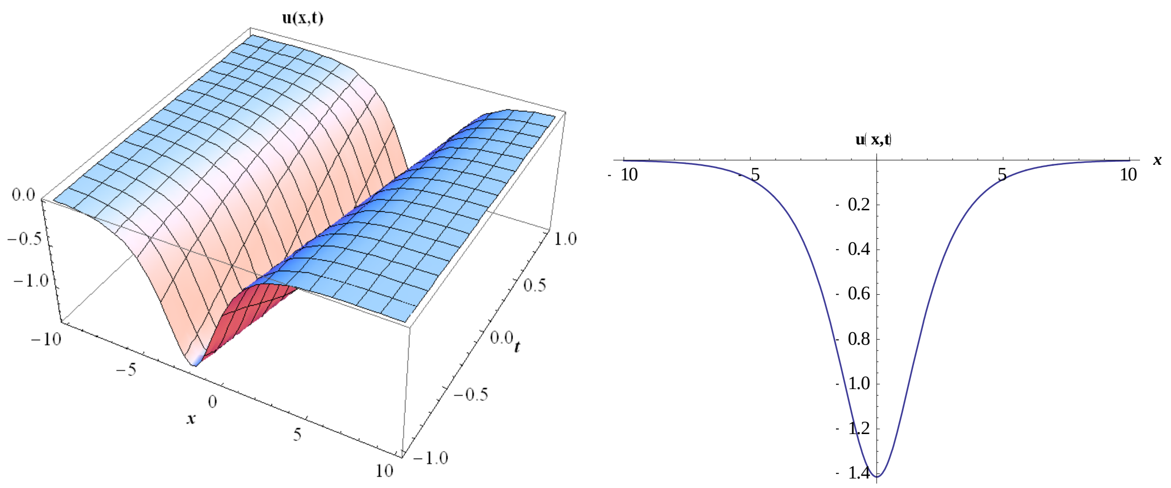


Figure.7 The three dimensional and two dimensional surfaces for Eq.(4.2.17) by considering the values $c = 2$, $b = 0.7, a = 3\ \mu = 0.7, -10 < x < 10,\ -1 < t < 1$ and for two dimensional $t = 0.002$.

Case-4

$$A_0 = 0, A_1 = \frac{-1}{\sqrt{a}}\sqrt{3(-1+c)}, B_1 = \frac{-1}{\sqrt{a}}\sqrt{3(1-c)}, b = \frac{2(-1+c)}{c\mu^2}. \tag{4.2.18}$$

If we get the coefficients of equation (4.2.18) for equation (3.9), we obtain the hyperbolic travelling wave solution for the mBBM equation as following;

$$u_4(x,t) = \frac{\sqrt{3}}{\sqrt{a}}\operatorname{sech}(\mu x - ct\mu)\left(\sqrt{1-c} + \sqrt{-1+c}\sinh(\mu x - ct\mu)\right). \tag{4.2.19}$$

Where a, c, μ are real constants and not zero.

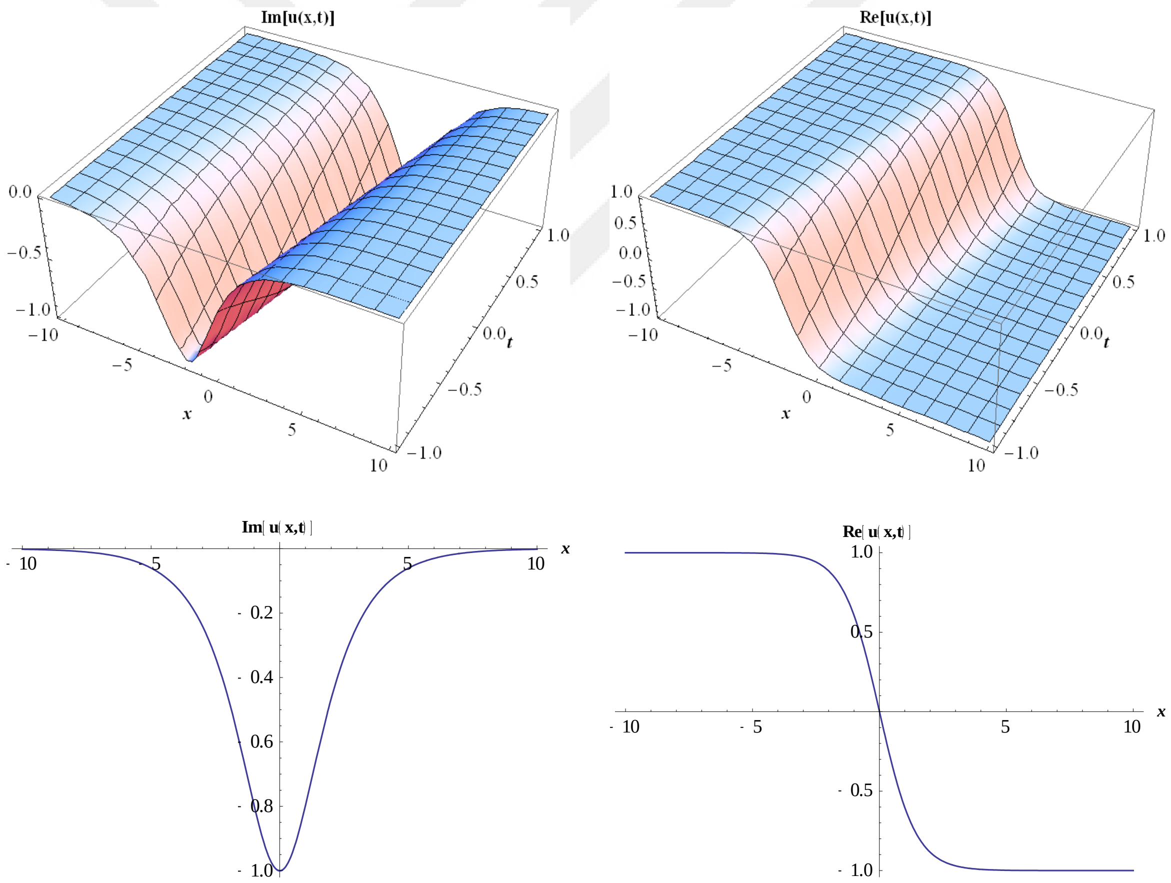


Figure.8 The three dimensional and two dimensional surfaces for Eq.(4.2.19) by considering the values $c = 2$, $b = 0.7, a = 3\ \mu = 0.7, -10 < x < 10,\ -1 < t < 1$ and for two dimensional $t = 0.002$.

4.3. Application of SGEM to the Equal width wave equation

Let’s consider the Equal Width Wave equation defined as following [22,25];

$$u_t + uu_x - \mu u_{xxt} = 0, \tag{4.3.1}$$

If we apply the following transformation,

$$u = u(x,t) = U(\xi), \xi = V(x - ct), \tag{4.3.2}$$

we can obtain the following nonlinear ordinary differential equation for equation (4.3.1)

$$-cVU' + UVU' + cV^3 \mu U''' = 0, \tag{4.3.3}$$

Where $u_x = VU', u_t = -cVU', u_{xxt} = -cV^3 U'''$. Taking integration of equation (4.3.3), we obtain the following equation;

$$-cVU + V\frac{U^2}{2} + cV^3 \mu U'' = 0, \tag{4.3.4}$$

Multiply 2 and divide by V to the equation (4.3.4), we get the following nonlinear ordinary differential equation for equation (4.3.1)

$$-2cU + U^2 + 2cV^2 \mu U'' = 0. \tag{4.3.5}$$

If we apply balance principle for equation (4.3.5) between U'' and U^2, we can obtain the following values of n.

$$n = 2. \tag{4.3.6}$$

When we put equation (4.3.6) in equation (3.10), we obtain following travelling wave solution for equation (4.3.4) as

$$U(w) = B_1 \sin(w) + A_1 \cos(w) + B_2 \cos(w)\sin(w) + A_2 \cos^2(w) + A_0. \tag{4.3.7}$$

If we consider equation (4.3.7) for second derivation of (4.3.5), we can write follows;

$$\begin{aligned} U''(w) &= B_1 \cos^2(w)\sin(w) - B_1 \sin^3(w) - 2A_1 \sin^2(w)\cos(w) \\ &+ B_2 \cos^3(w)\sin(w) - 5B_2 \sin^3(w)\cos(w) - 4A_2 \cos^2(w)\sin^2(w) \\ &+ 2A_2 \sin^4(w). \end{aligned} \tag{4.3.8}$$

If we consider equations (4.3.7), (4.3.8) for equation (4.3.5), we can obtain the following equation;

$$
\begin{aligned}
&-2c[B_1\sin\left(w\right)+A_1\cos\left(w\right)+B_2\cos(w)\sin(w)+A_2\cos^2(w)+A_0] \\
&+[B_1\sin\left(w\right)+A_1\cos\left(w\right)+B_2\cos(w)\sin(w)+A_2\cos^2(w)+A_0]^2 \\
&+2c\mu V^2[B_1\cos^2\left(w\right)\sin\left(w\right)-B_1\sin^3\left(w\right)-2A_1\sin^2\left(w\right)\cos\left(w\right) \\
&+B_2\cos^3(w)\sin(w)-5B_2\sin^3(w)\cos(w)-4A_2\cos^2(w)\sin^2(w) \\
&+2A_2\sin^4(w)]=0.
\end{aligned}
\tag{4.3.9}
$$

We can rewrite equation (4.3.9) as following;

$$
\begin{aligned}
&-2cB_1\sin\left(w\right)-2cA_1\cos\left(w\right)-2cB_2\cos(w)\sin(w)-2cA_2\cos^2(w) \\
&-2cA_0+B_1^2\sin^2\left(w\right)+A_1^2\cos^2(w)+B_2^2\cos^2(w)\sin^2(w) \\
&+A_2^2\cos^4(w)+A_0^2+2B_1A_1\cos(w)\sin(w)+2B_1B_2\sin^2\left(w\right)\cos\left(w\right) \\
&+2B_1A_2\cos^2(w)\sin(w)+2B_1A_0\sin(w)+2B_2A_1\cos^2(w)\sin(w) \\
&+2A_2A_1\cos^3(w)+2A_1A_0\cos(w)+2B_2A_2\cos^3(w)\sin(w) \\
&+2B_2A_0\cos(w)\sin(w)+2A_2A_0\cos^2(w)-8c\mu V^2A_2\cos^2(w)\sin^2(w) \\
&+4c\mu V^2A_2\sin^4(w)+2c\mu V^2B_1\cos^2\left(w\right)\sin\left(w\right)-2c\mu V^2B_1\sin^3\left(w\right) \\
&+2c\mu V^2B_2\cos^3(w)\sin(w)-10c\mu V^2B_2\sin^3(w)\cos(w) \\
&-4c\mu V^2A_1\sin^2\left(w\right)\cos\left(w\right)=0.
\end{aligned}
\tag{4.3.10}
$$

After then, when we sort out equation (4.3.10) according to trigonometric functions, we can type follows;

$$
\begin{aligned}
&constant &&: -2cA_0 + A_0^2 + B_1^2 + 4c\mu V^2 A_2 = 0,\\
&\sin(w) &&: 2B_1A_0 - 2cB_1 - 2c\mu V^2 B_1 = 0,\\
&\cos(w) &&: -2cA_1 + 2A_1A_0 + 2A_2A_1 = 0,\\
&\cos(w)\sin(w) &&: -2cB_2 + 2B_1A_1 + 2B_2A_0 - 10c\mu V^2 B_2 = 0,\\
&\cos^2(w) &&: -2cA_2 + A_1^2 + 2A_2A_0 + A_2^2 - B_1^2 - 4c\mu V^2 A_2 = 0,\\
&\sin^2(w)\cos(w) &&: 2B_1B_2 - 4c\mu V^2 A_1 - 2A_2A_1 + 2c\mu V^2 B_1 = 0,\\
&\cos^2(w)\sin(w) &&: 2B_1A_2 + 2B_2A_1 + 2c\mu V^2 B_1 = 0,\\
&\cos^2(w)\sin^2(w) &&: B_2^2 - 12c\mu V^2 A_2 - A_2^2 = 0,\\
&\cos^3(w)\sin(w) &&: 2B_2A_2 + 12c\mu V^2 B_2 = 0.
\end{aligned}
\tag{4.3.11}
$$

If we solve this system by using the programming (Wolfram Mathematica), we can find the following coefficients which give us many different travelling wave solutions for equation (4.3.1);

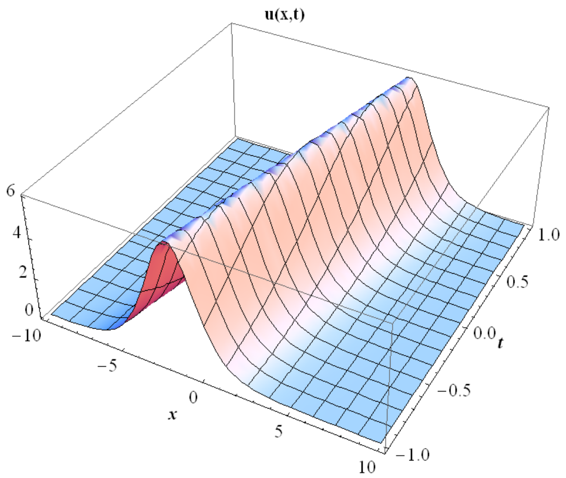
Case-1

$$A_0 = 3c, A_1 = 0, B_1 = 0, A_2 = -3c, B_2 = 0, V = \frac{1}{2\sqrt{\mu}}. \tag{4.3.12}$$

If we get the coefficients of equation (4.3.12) for equation (3.9), we can get the travelling wave solution for the the Equal Width Wave equation as following;

$$u_1(x,t) = 3c\,\mathrm{sech}\left(\frac{-ct+x}{2\sqrt{\mu}}\right). \tag{4.3.13}$$

Where c,μ are real constants and non-zero.



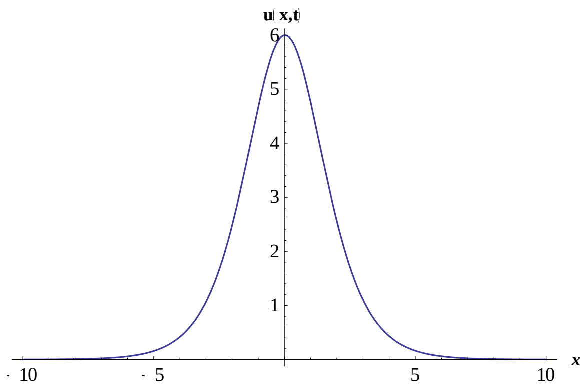


Figure.9 The three dimensional and two dimensional surfaces for Eq.(4.3.13) by considering the values $c = 2,\ \mu = 1, -10 < x < 10,\ -1 < t < 1$ and for two dimensional $t = 0.002$.

Case-2

$$A_0 = -c, A_1 = 0, B_1 = 0, A_2 = 3c, B_2 = 0, \mu = -\frac{1}{V^2}. \tag{4.3.14}$$

If we get the coefficients of equation (4.3.14) for equation (3.9), we can get the travelling wave solution for the the Equal Width Wave equation as following;

$$u_2(x,t) = c\left(-1 + 3\tanh\left[V(-ct + x)\right]^2\right). \tag{4.3.15}$$

Where c, V are real constants and not zero.

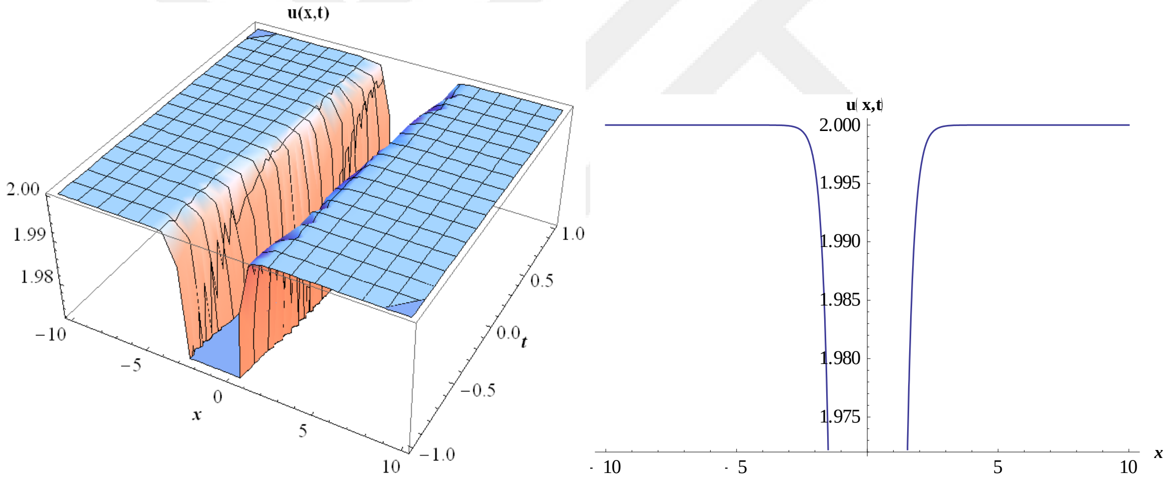


Figure.10 The three dimensional and two dimensional surfaces for Eq.(4.3.15) by considering the values $c = 1,\ V = 2, -10 < x < 10,\ -1 < t < 1$ and for two dimensional $t = 0.002$.

Case-3

$$A_0 = -c, A_1 = 0, B_1 = 0, A_2 = 3c, B_2 = 0, V = -\frac{i}{2\sqrt{\mu}}. \tag{4.3.16}$$

If we get the coefficients of equation (4.3.16) for equation (3.9), we can get the travelling wave solution for the the Equal Width Wave equation as following;

$$u_3(x,t) = c\left(-1 - 3\tan\left[\frac{(-ct+x)}{2\sqrt{\mu}}\right]^2\right). \tag{4.3.17}$$

Where c, μ are real constants and not zero.

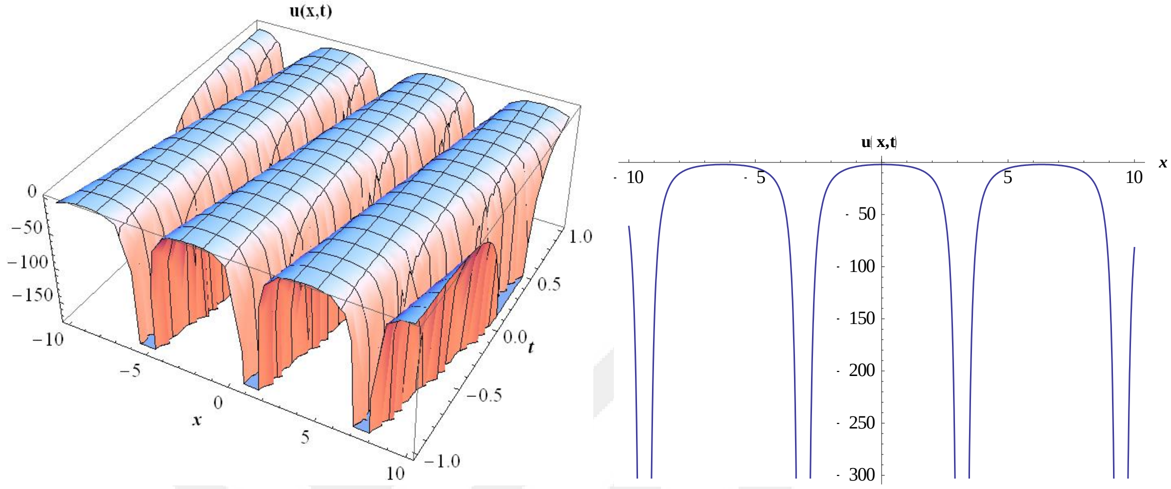


Figure.11 The three dimensional and two dimensional surfaces for Eq.(4.3.17) by considering the values $c = 2$, $\mu = 1, -10 < x < 10$, $-1 < t < 1$ and for two dimensional $t = 0.002$.

Case-4

$$A_0 = 6c, A_1 = 0, B_1 = 0, A_2 = -6c, B_2 = -6ic, V = \frac{1}{\sqrt{\mu}}. \tag{4.3.18}$$

If we get the coefficients of equation (4.3.18) for equation (3.9), we can get the travelling wave solution for the the Equal Width Wave equation as following;

$$u_4(x,t) = 6c\sec\left[\frac{(-ct+x)}{\sqrt{\mu}}\right]\left(\operatorname{sech}\left[\frac{(-ct+x)}{\sqrt{\mu}}\right] - i\tanh\left[\frac{(-ct+x)}{\sqrt{\mu}}\right]\right). \tag{4.3.19}$$

Where c, μ are real constants and not zero.

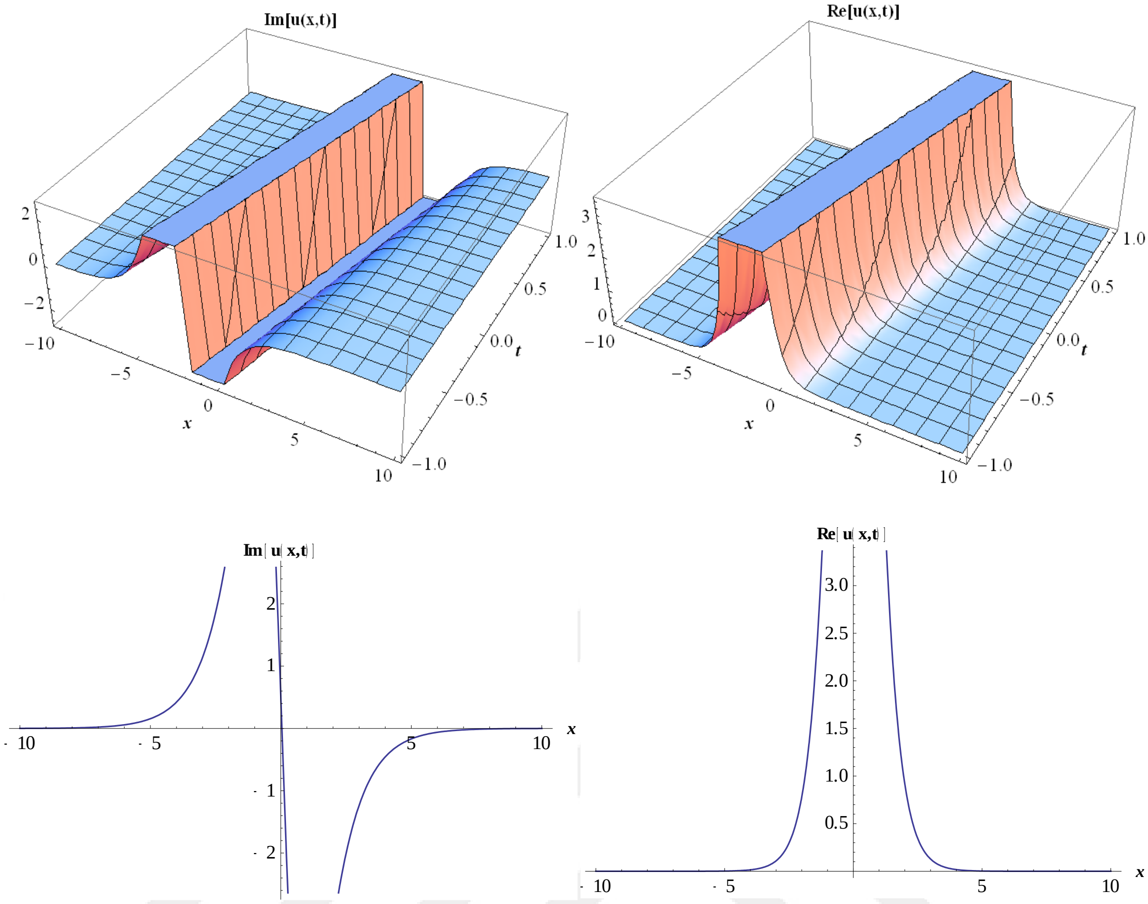


Figure.12 The three dimensional and two dimensional surfaces for Eq.(4.3.19) by considering the values $c = 2$, $\mu = 1, -10 < x < 10$, $-1 < t < 1$ and for two dimensional $t = 0.002$.

4.4. Application of SGEM to the modified Korteweg–de Vries (mKdV) equation

Let us consider the mKdV equation defined as following [24];

$$u_t - u^2 u_x + \delta u_{xxx} = 0. \tag{4.4.1}$$

If we apply the following transformation,

$$u = u(x,t) = U(\xi)\, and\ \xi = \mu(x - ct), \tag{4.4.2}$$

we can obtain the following nonlinear ordinary differential equation for equation (4.4.1)

$$-c\mu U' - U^2 \mu U' + \delta\mu^3 U''' = 0. \tag{4.4.3}$$

Where $u_t = -c\mu U', u_x = \mu U', u_{xxx} = \mu^3 U'''.$ Taking integration of equation (4.4.3), we obtain the following equation;

$$-c\mu U - \mu\frac{U^3}{3} + \delta\mu^3 U'' = 0. \tag{4.4.4}$$

Multiply 3 and divide by μ to the equation (4.4.4), we get the following nonlinear ordinary differential equation for equation (4.4.1)

$$-3cU - U^3 + 3\delta\mu^2 U'' = 0. \tag{4.4.5}$$

If we apply balance principle for equation (4.4.5) between U'' and U^3, we can obtain the following values of n.

$$n = 1. \tag{4.4.6}$$

When we put equation (4.4.6) in equation (3.10), we obtain following travelling wave solution for equation (4.4.4) as

$$U(w) = B_1 \sin(w) + A_1 \cos(w) + A_0. \tag{4.4.7}$$

If we consider equation (4.4.7) for second derivation of (4.4.5), we can write follows;

$$U''(w) = B_1 \cos^2(w)\sin(w) - B_1 \sin^3(w) - 2A_1 \sin^2(w)\cos(w). \tag{4.4.8}$$

If we consider the equations (4.4.7), (4.4.8) for equation (4.4.5), we can obtain the following equation;

$$-3c\left(B_1\sin(w)+A_1\cos(w)+A_0\right)-\left(B_1\sin(w)+A_1\cos(w)+A_0\right)^3$$
$$+3\delta\mu^2[B_1\cos^2(w)\sin(w)-B_1\sin^3(w)-2A_1\sin^2(w)\cos(w)]=0. \tag{4.4.9}$$

We can rewrite equation (4.4.9) as following;

$$\begin{aligned}
&-3cB_1\sin(w)-3cA_1\cos(w)-3cA_0-B_1^3\sin^3(w)-A_1^3\cos^3(w)\\
&-A_0^3-6B_1A_1A_0\sin(w)\cos(w)-3B_1^2A_1\sin^2(w)\cos(w)\\
&-3B_1^2A_0\sin^2(w)-3A_1^2B_1\cos^2(w)\sin(w)-3A_1^2A_0\cos^2(w)\\
&-3A_0^2B_1\sin(w)-3A_0^2A_1\cos(w)+3\delta\mu^2B_1\cos^2(w)\sin(w)\\
&-3\delta\mu^2B_1\sin^3(w)-6\delta\mu^2A_1\sin^2(w)\cos(w)=0.
\end{aligned} \tag{4.4.10}$$

After then, when we sort out equation (4.4.10) according to trigonometric functions, we can type follows;

$$\begin{aligned}
&constant &&: -3cA_0-A_0^3-3B_1^2A_0=0,\\
&\sin(w) &&: -3cB_1-3A_0^2B_1=0,\\
&\cos(w) &&: -3cA_1-3A_0^2A_1-A_1^3=0,\\
&\sin(w)\cos(w) &&: -6B_1A_1A_0=0,\\
&\cos^2(w) &&: -3A_1^2A_0+3B_1^2A_0=0,\\
&\sin^2(w)\cos(w) &&: -3B_1^2A_1-6\delta\mu^2A_1+A_1^3=0,\\
&\cos^2(w)\sin(w) &&: -3A_1^2B_1+3\delta\mu^2B_1=0,\\
&\sin^3(w) &&: -B_1^3-3\delta\mu^2B_1=0.
\end{aligned} \tag{4.4.11}$$

If we solve this system by using the programming (Wolfram Mathematica), we can find the following coefficients which give us many different travelling wave solutions for equation (4.4.1);

Case-1

$$A_0=0, A_1=-\sqrt{6}\sqrt{\delta}, B_1=0, c=-2\delta\mu^2. \tag{4.4.12}$$

If we get the coefficients of equation (4.4.12) for equation (4.9), we can get the travelling wave solution for the mKdV equation as following;

$$u_1(x,t)=-\sqrt{6}\sqrt{\delta}\mu\tanh\left(\mu\left(x+2t\delta\mu^2\right)\right). \tag{4.4.13}$$

Where δ, μ are real constants and not zero.

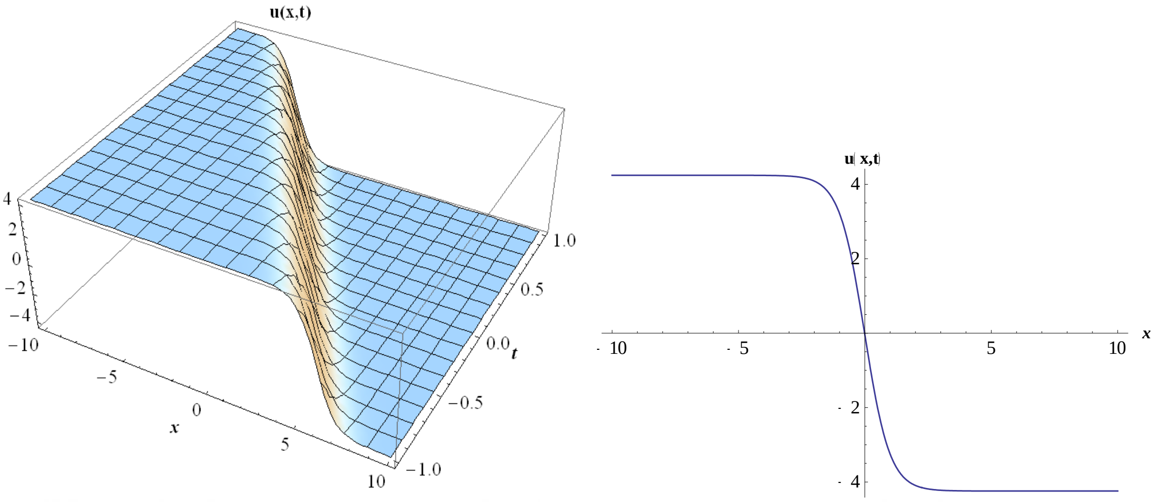


Figure.13 The three dimensional and two dimensional surfaces for Eq.(4.4.13) by considering the values $\delta = 3,\ \mu = 1, -10 < x < 10,\ -1 < t < 1$ and for two dimensional $t = 0.002$.

Case-2

$$A_0 = 0, A_1 = \sqrt{6}\sqrt{\delta}, B_1 = 0, c = -2\delta\mu^2. \tag{4.4.14}$$

If we get the coefficients of equation (4.4.14) for equation (3.9), we can get the travelling wave solution for the mKdV equation as following;

$$u_2(x,t) = \sqrt{6}\sqrt{\delta}\mu \tanh\left(\mu\left(x + 2t\delta\mu^2\right)\right). \tag{4.4.15}$$

Where δ, μ are real constants and non-zero.

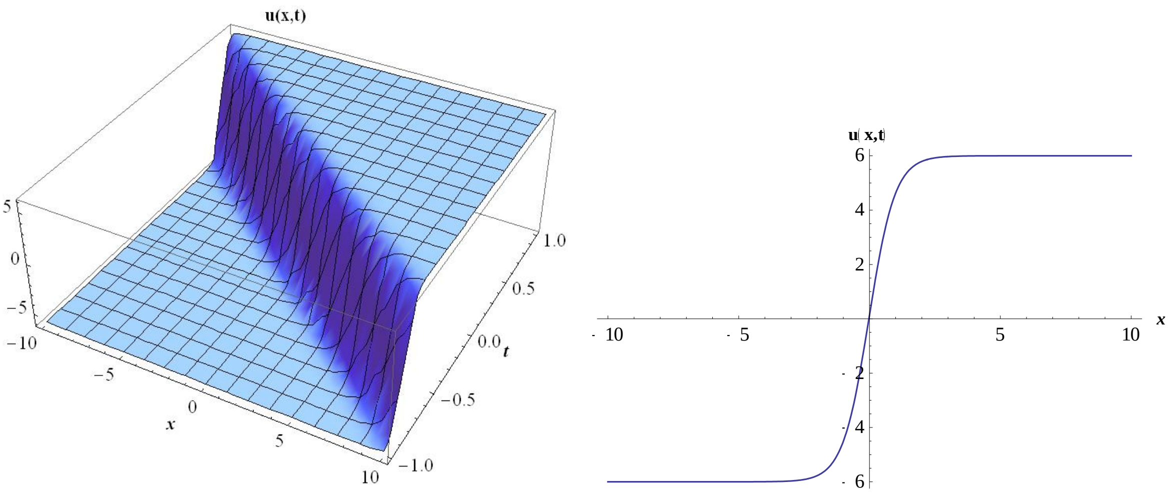


Figure.14 The three dimensional and two dimensional surfaces for Eq.(4.4.15) by considering the values $\delta = 6$, $\mu = 1, -10 < x < 10,\ -1 < t < 1$ and for two dimensional $t = 0.002$.

Case-3

$$A_0 = 0, A_1 = -i\sqrt{3}\sqrt{c}, B_1 = 0, \mu = \frac{i\sqrt{c}}{\sqrt{2}\sqrt{\delta}}. \tag{4.4.16}$$

If we get the coefficients of equation (4.4.16) for equation (3.9), we can get the travelling wave solution for the mKdV equation as following;

$$u_3(x,t) = \sqrt{3}\sqrt{c}\tan\left(\frac{\sqrt{c}(-ct+x)}{\sqrt{2}\sqrt{\delta}}\right). \tag{4.4.17}$$

Where δ is real constant and non-zero.

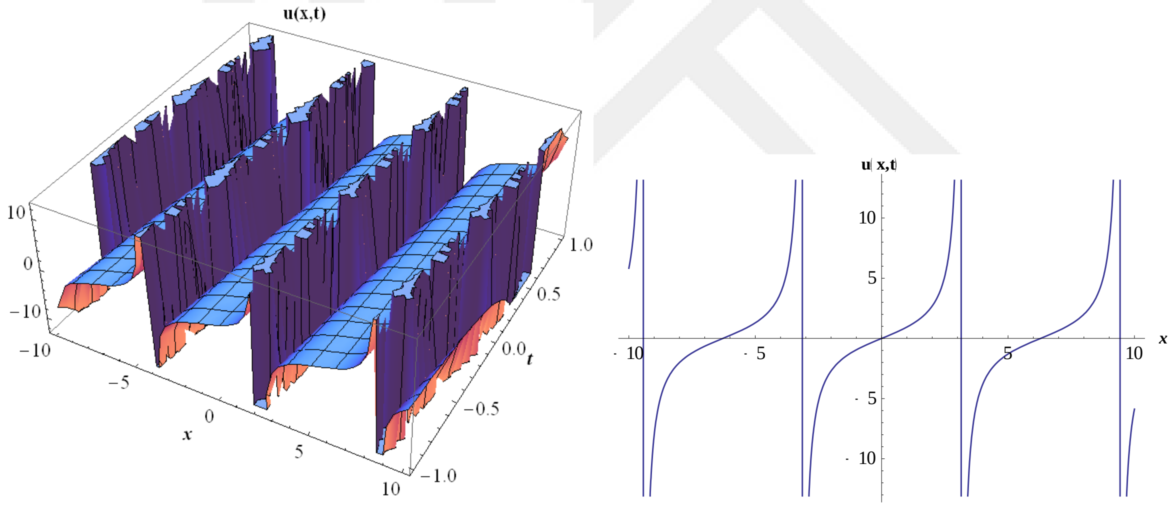


Figure.15 The three dimensional and two dimensional surfaces for Eq.(4.4.17) by the values $\delta = 2$, $c = 1, -10 < x < 10,\ -1 < t < 1$ and for two dimensional $t = 0.002$.

Case-4

$$A_0 = 0, A_1 = i\sqrt{3}\sqrt{c}, B_1 = 0, \mu = \frac{i\sqrt{c}}{\sqrt{2}\sqrt{\delta}}. \tag{4.4.18}$$

If we get the coefficients of equation (4.4.18) for equation (3.9), we can get the travelling wave solution for the mKdV equation as following;

$$u_4(x,t) = -\sqrt{3}\sqrt{c}\tan\left(\frac{\sqrt{c}\,(-ct+x)}{\sqrt{2}\sqrt{\delta}}\right). \tag{4.4.19}$$

Where c,δ are real constants and not zero.

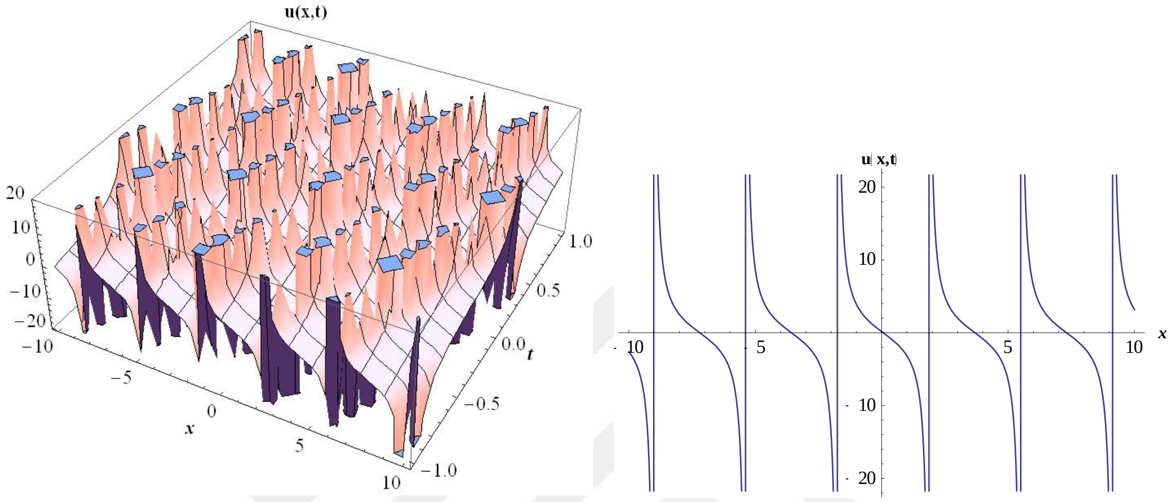


Figure.16 The three dimensional and two dimensional surfaces for Eq.(4.4.19) by the values $\delta=2$, $c=3, -10<x<10$, $-1<t<1$ and for two dimensional $t=0.002$.

Note: check all the coefficient and the solutions in the Mathematica 9 program to make sure that you clear all error (if any), and also check the values for all the constant in the plotted graphics, this should be applied to all the solved models.

5. CONCLUSIONS

In this study, the general structures of the Sine-Gordon Expansion Method have been presented. After then, the travelling wave solutions like hyperbolic function solutions to the equation (2+1) dimensional Zakharov-Kuznetsov, the modified Benjamin–Bona–Mahony equation, the Equal width wave equation, the modified Korteweg–de Vries equation have been obtained by using the Sine-Gordon expansion method. Furthermore, the two- and three-dimensional surfaces for models considered in this study have been plotted with the aid of symbolic computation programming language.

It has been observed that these solutions have verified the modified Korteweg–de Vries equation, the Equal Width Wave equation, the Zakharov-Kuznetsov equation, the modified Benjamin–Bona–Mahony equation, with using program (Wolfram Mathematica).

To the best of our insight, the application of Sine-Gordon expansion method to these mathematical models have not been submitted to the literature in advance.

6. REFERENCES

[1] **Mothibi, D.M. and Khalique, C.M.,** 2015. Conservation Laws and Exact Solutions of a Generalized Zakharov-Kuznetsov Equation, *Symmetry*. **7**, 949-961.

[2**] Munro, S. and Parkes E.J.,** 1999. The Derivation of a Modified Zakharov-Kuznetsov Equation and the Stability of its Solution, *Journal of Plasma Physics*. **62** 305-317.

[3] **Abbasbandy, S. and Shirzadi, A.,** 2010. "The first integral method for modified Benjamin-Bona-Mahony equation," *Communications in Nonlinear Science and Numerical Simulation,* vol. **15**, no. 7, pp.1759–1764.

[4] **Hong, B. and Lu, D.,** 2008. "New exact solutions for the generalized BBM and Burgers-BBM equations," *World Journal of Modelling and Simulation*, vol. **4**, pp. 243–249.

[5] **Morrison, P. J. and Meiss, J. D.** 1984 Physical D: *Nonlinear Phenomena* 30 324

[6] **George, A.** 1994 Solving Frontier Problems of Physics: Adomain Decomposition Method (*Boston: Kluwer Academic Publishers*)

[7] **Shigeki Matsutani and Hideo Tsuru**, 1991. *Journal of the Physical Society of Japan*, **60**, 3640–3644 .

[8] **Hiroaki Ono,** 1992. Soliton Fission in a harmonic Lattices with Reflection Less Inhomogeneity, *Journal of the Physical Society of Japan*, **61**, 4336–4343

[9] **Ralph, E. A. and Pratt L.,** 1994. Predicting eddy detachment for an equivalent bar tropic thin jet, *J. Nonlinear Sci*.**4**, 355–374.

[10] **Komatsu, T. S. and Sasa, S.-I.,** 1995. *Phys. Rev*. E, **52**, 5574–5582 (1995); arXiv: patt- sol/9506002v1.

[11] **Ge, H. X., Dai, S. Q., Xue, Y and Dong, L. Y.,** 2005. Stabilization analysis and modified Korteweg–de Vries equation in a cooperative driving system Phys. Rev. E, 71, 066119.

[12] **Schief, W.K.** 1995. an infinite hierarchy of symmetries associated with hyperbolic surfaces, *Nonlinearity* **8** 1-9.

[13] **Lonngren, K.E.** 1998. Ion acoustic soliton experiments in a plasma. *Opt. Quant. Electron*, 30: 615–630.

[14] **Khater, A. H. and El-Kalaawy, O. H. and Callebaut, D. K.,** 1998. Backlund transformations and exact solutions for Alfven solitons in a relativistic electron-positron plasma **58**, 545–548 (1998).

[15] **Agop, M. and Cojocaru, V.,** 1998. Oscillation modes of slag-metallic bath inter-face, Mater. *Trans. JIM* **39**, 668-671.

[16] **Ziegler, V. Dinkel, J. Setzer, C. and Lonngren, K. E.,** 2001. *Chaos Solitons Fractals*, **12**, 1719–1728 (2001).

[17] **Wazwaz, Abdul-Majid, 2009.** Partial Differential Equations and Solitary Waves Theory, **4-15**.

[18] **Chuntao Yan, 1996.** a simple transformation for nonlinear waves, *Physics Letters* A, **224**, 77-84, 1996.

[19] **Zhenya Yan, Hongqing Zhang, 1999.** New explicit and exact travelling wave solutions for a system of variant Boussinesq equations in mathematical physics, *Physics Letters A*, **252**, 291-296.

[20] **Yan Zhen-Ya, Zhang Hong-Oing, Fan En-Gui**, 1999. New explicit and travelling wave solutions for a class of nonlinear evaluation equations, *Acta Physical Sinica*, **48**(1), 1-5.

[21] **Shafiqul Islam, Kamruzzaman Khan, Md., Ali Akbar, M. and Antonio Mastroberardino,** a note on improved F-expansion method combined with Riccati equation applied to nonlinear evolution equations, *Royal Society Open Science*, http://dx.doi.org/10.1098/rsos.140038.

[22] **Arora, Rajan, Junaid Siddiqui, Md. and Singh, V. P.** 2012. Solutions of Inviscid Burgers' and Equal Width Wave Equations by RDTM, *International Journal of Applied Physics and Mathematics,* Vol. **2**, No. 3, May.

[23] **Baskonus,H. M., Sulaiman, T. A., Bulut H.,** 2017. On the novel wave behaviors the coupled nonlinear Maccari's system with complex structure, *Optik*, **131**, 1036-1043.

[24] **Zun-Tao, FU., Shi-Kuo, LIU., and Shi-Da, LIU.,** 2002. School of Physics, Peking University, Beijing 100871.

[25] **Gardner, L. R. T. and Gardner, G. A.** 1992. "Solitary waves of the equal width wave equation*," Journal of Computational Physics*, vol. **101**, pp. 218-223.

[26] **Bleecker, D. and Csordas, G.,** 1995. Basic Partial Differential Equations, Chapman and Hall, New York.

[27] **David R. Kincaid, Ward Cheney, E.,** 1991. Numerical Analysis: Mathematics of Scientific Computing, 487-488.

[28] **Hendra Adiwidjaja**, 1995. On the korteweg de Vries equation, December 9.

[29] **Baskonus, H. M.** 2016. New acoustic wave behaviors to the Davey–Stewartson equation with power-law nonlinearity arising in fluid dynamics, *Nonlinear Dynamics*, **86(1),**177-183.

[30] **Baskonus, H. M., Bulut, H. and Atangana, A.** 2016. On the complex and hyperbolic structures of the longitudinal wave equation in a magneto-electro-elastic circular rod, *Smart Materials and Structures*. **25 (3)**,035022.