

REPUBLIC OF TURKEY
FIRAT UNIVERSITY
THE GRADUATE SCHOOL OF NATURAL AND APPLIED SCIENCE



**SOME NEW APPLICATIONS
OF THE
RESIDUAL POWER SERIES METHOD**
HARIVAN RAMADHAN NABI
(142121102)

Master Thesis
Department: Mathematics
Program: Applied Mathematics
Supervisor: Prof. Dr. Mustafa İNÇ
DECEMBER-2016

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Harivan Ramdhan Nabi

SUMMARY

SOME NEW APPLICATIONS OF THE RESIDUAL POWER SERIES METHOD

This Thesis includes five chapters.

In the chapter one, we give some of the information in this topic.

In the chapter two, some definitions and theorems which used in the main section are given.

In chapter three, the formula of residual power series method is given. This section is the main part of the study. We explain how the residual power series method is applied to the problems.

In chapter four, in order to find out series solution for the CBEs with initial conditions, some examples of nonlinear Burger's equations have been illustrated. Two and three dimensional graphs related to the obtained solutions were drawn with Mathematica and Maple programs. We give conclusion and discussion in the final chapter.

Keywords: Burgers' and coupled Burgers' Eqs.; Residual power series method; Nonlinear partial differential equation; Numerical solution.

ÖZET

RESİDUAL GÜÇ SERİSİ YÖNTEMİNİN BAZI YENİ UYGULAMALARI

Bu tez beş bölümden oluşmaktadır.

Birinci bölümde Burgers, coupled Burgers ve residual kuvvet serisi yöntemiyle ilgili bazı bilgiler verilmiştir.

İkinci bölümde bazı temel tanım ve teoremler verildi.

Üçüncü bölümde residual kuvvet serisi metodunun formülü verildi. Çalışmanın esas kısmını oluşturan bu bölümde residual kuvvet serisi metodunun nasıl elde edildiği bahsedilen denklemlere nasıl uygulandığı izah edilmiştir.

Dördüncü bölümde başlangıç şartlı coupled Burgers' denklemlerinin seri çözümlerini bulmak için bazı örnekler verildi. Elde edilen çözümlerle ilgili iki ve üç boyutlu resimler Matematika ve Maple programı ile çizilmiştir. Son bölümde ise sonuç ve tartışmaya yer verilmiştir.

LIST OF SYMBOLS

RPSM: Residual power series method

DADM: Discrete Adomian decomposition method

CBE: Coupled Burgers' equation

KdV: Korteweg-de Vries equation

IFDM: Implicit finite difference method

BVP: Boundary value problem



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1- Introduction

In recent years, it made a lot of efforts to calculate the accuracy and efficiency of various numerical schemes to offset the CBEs with different values of kinematic viscosity. It has been used widely Burgers' equation in modeling the various phenomena in science and engineering with a wide range of implementations, containing gas dynamics, and the flow of traffic, and the propagation of waves in hydrodynamics and acoustics, etc. It was the first initiative to investigate the analytical one-dimensional Burgers' equation by Bateman [1].

The goal of this work is to implement the residual power series method (RPSM) to the one-dimensional Burgers', the coupled Burgers' and inhomogeneous Burgers' equations. One-dimensional nonlinear Burgers' equation in the following two different forms:

$$\begin{cases} u_t - uu_x - \nu u_{xx} = 0 \\ u_t + uu_x - \nu u_{xx} = 0 \end{cases}; \quad x \in \Delta, \quad t > 0 \quad (1.1)$$

where $\Delta = \{x \in \mathbb{R} : \mathcal{A} \leq x \leq \mathcal{B}\}$, together with initial condition:

$$u(x, 0) = f(x) \quad x \in \Delta \quad (1.2)$$

two-dimensional nonlinear coupled Burgers' system:

$$\begin{cases} u_t + uu_x + \nu u_y = \frac{1}{R}(u_{xx} - u_{yy}) \\ v_t + uv_x + \nu v_y = \frac{1}{R}(v_{xx} - v_{yy}) \end{cases}, \quad (x, y) \in \Delta_1, \quad t > 0 \quad (1.3)$$

with the following initial conditions

$$\begin{cases} u(x, y, 0) = f(x, y) \\ v(x, y, 0) = g(x, y) \end{cases}, \quad (x, y) \in \Delta_1 \quad (1.4)$$

where $\Delta_1 = \{(x, y) \in \mathbb{R} : \mathcal{A} \leq x \leq \mathcal{B}, \mathcal{C} \leq y \leq \mathcal{D}\}$ is the computational domain; and R is the Reynolds number; and ν is arbitrary constant.

Inhomogeneous Burgers' equation is presented by:

$$u_t + u_{xx} - uu_x = f(x, t) \quad (1.5)$$

Subject to initial condition

$$u(x, 0) = 0. \quad (1.6)$$

where $0 \leq t \leq 1, 0 \leq x \leq 1$ and $f(x, t)$ is known function.

Burgers' equation has already been resolved using a variety of analytical and numerical schemes, for example, finite difference method [2], modified cubic B-splines collocation method [3], inverse scattering method [4], the sine–cosine method [5,6], to nonlinear polycrystalline solids [7], Benton and Platzman [8] exact solutions that surveyed of the one-dimensional Burgers' equation. Generalized differential quadrature method [9], an implicit logarithmic finite difference method [10], Cole [11] and Hopf [12] have shown independent of the initial conditions the one-dimensional Burgers' equation, can turn into a homogeneous linear heat that can be solved analytically equation. Quadratic B-spline finite elements [13], Variational iteration method [14], Galerkin quadratic B-spline finite element method [15], Fourier Pseudo spectral method [16], Darboux transformation [17], in [18], Xie et al. gave a numerical method for Burgers' equation by utilizing piecewise quadratic polynomial basis functions and the reproducing kernel function. Hermite collocation method in [19], Recently an extended tanh-function technique and computation symbolic are given in [20] for investigating the new coupled modified KDV Eqs. to acquire four kinds soliton solutions.

The coupled Burgers' system is obtained by Esipov [21]. It is a basic model of the evolution of sedimentation or concentrations scaled the size of two types of molecules in liquids or colloidal suspension, under the influence of gravity [22].

There are several approximate analytical methods to solution of problems in the literature, such as the variational iteration method [23–26] was implemented to autonomous ordinary differential equations in [27], to nonlinear polycrystalline solids [28], and other fields. And homotopy analysis method proposed by Liao [29, 30].

At the present study, in the power series method we researched the residual concept to get a method RPSM for describing values of coefficients of the series solutions for fuzzy differential equations [31], in [32] the generalized Lane-Emden equation, it has been implemented in the numerical solution, in [33] the solution of composite and non-composite fractional differential equations, in create and foretell the solitary pattern solutions for nonlinear time-fractional dispersive PDEs [34], in the numerical solution of higher-order regular differential equations [35], in approach solution of the nonlinear fractional KdV-Burgers' equation [36], in foretell and give the multiplicity of solutions to BVPs of fractional order [37], and in the numerical solutions of linear non-homogeneous PDEs of fractional order [38]. The residual power series method is efficient to utilize for solving linear and nonlinear of (CBEs) without perturbation, linearization, or discretization. This method

calculate the coefficient of the PS by a chain of linear equations of m -variable, where m is no. of equations. Besides, the residual power series method does not get to any modification while switching from the first order to the higher order; thus, the technique can be implemented directly to the examples by selecting a convenient initial guess approximation.

The outline of the remainder of this work is as follows. In the next section, we give some basic definitions and theorems. In Section 3, the way of constructing the RPSM for solving the Burgers' and coupled Burgers' equations, and in Section 4, the proposed method is applied to problems of the nonlinear Burgers' equations. The conclusion and discussion is given in the final part, Section 5.



2. Basic Definitions and Theorems

2.1 Partial Differential Equations (PDEs)

There is a lot independent variable $x, y, \dots$ for describing a feature of partial differential equation (PDF). There is dependent variable which is an unknown function containing independent variables, say $u(x, y, \dots)$. Define its derivative by prime or subscript. We have

$$F(x, y, u(x, y), u_x(x, y), u_y(x, y)) = F(x, y, u, u_x, u_y) = 0. \quad (2.1)$$

Eq. (2.1) is the first order PDE in two independent variables. The most general second-order partial differential equation in two independent variables is given as:

$$F(x, y, u, u_x, u_y, u_{xx}, u_{yy}) = 0. \quad (2.2)$$

Let consider the equation has $\mathcal{L}u = 0$, here $\mathcal{L}$ is a linear operator. If v is any function then $\mathcal{L}v$ is a novel function. $\mathcal{L} = \partial/\partial x$ is the linear operator that gets v into its partial derivative v_y . The linearity is given as:

$$\mathcal{L}(u + v) = \mathcal{L}u + \mathcal{L}v \quad \mathcal{L}(cu) = c\mathcal{L}u \quad (2.3)$$

for any functions u, v and any constant c .

$$\mathcal{L}u = 0 \quad (2.4)$$

If $\mathcal{L}$ is linear operator then Eq. (2.4) is linear.

$$\mathcal{L}u = g, \quad (2.5)$$

where $g \neq 0$ is a function. For example, Eq. (2.6) is said to be inhomogeneous linear.

$$(\cos xy^2)u_x - y^2u_y = \tan(x^2 + y^2). \quad (2.6)$$

2.2 Order of a PDE

The order of a partial differential equation is the order of the highest derivative that we get in the equation. The following equations

$$\left. \begin{aligned} u_x - u_y &= 0, \\ u_{xx} - u_t &= 0, \\ u_y - uu_{xxx} &= 0, \end{aligned} \right\} \quad (2.7)$$

Eq. (2.7) are partial differential equations of first order, second order, and third order.

2.3 Linear and Nonlinear PDEs

If the partial differential equations satisfied the following two conditions is called linear otherwise the equation is called nonlinear:

- (1) The power of the dependent variable and each partial derivative involved in the equation should be one.
- (2) The coefficients of the dependent variable and any partial derivative must be constants or independent variables.

2.4 Initial Conditions (I.C.) and Boundary Conditions (B.C.):

An I.C. refers to the physical state at specific time t_0 . The initial condition for the diffusion equation is given as:

$$u(x, t_0) = \phi(x), \quad (2.8)$$

where $\phi(x) = \phi(x, y, z)$ is a function. $\phi(x)$ is the initial concentration for a diffusing substance. For heat flow, $\phi(x)$ is the initial temperature. For the Schrödinger equation, too, (2.8) is usual an I.C.

Wave equation can be written as a pair of initial conditions

$$u(x, t_0) = \phi(x) \quad \text{and} \quad \frac{\partial u}{\partial t} = \psi(x), \quad (2.9)$$

where $\phi(x)$ describe the initial position and $\psi(x)$ describe the initial velocity.

The three most significant types of B.C. are given as:

(D) u is determined ("Dirichlet condition")

(N) The $\partial u / \partial n$ is determined ("Neumann condition")

(R) $\frac{\partial u}{\partial n} + au$ is determined ("Robin condition")

where a is a function of x, y, z, t . We give (D), (N), and (R) as equations. For example, (N) is written as the form

$$\frac{\partial u}{\partial n} = g(x, t) \quad (2.10)$$

where g is the boundary datum. If the determined function $g(x, t)$ vanishes, any of these B.C. is called homogeneous. Conversely, it is said to inhomogeneous. As usual, $n =$

(n_1, n_2, n_3) shows the unit normal vector on ∂D , which points outward from D (see figure 1). Also $\partial u / \partial n = \nabla u \cdot \mathbf{n}$ defines the directional derivative of u toward the outside normal direction.

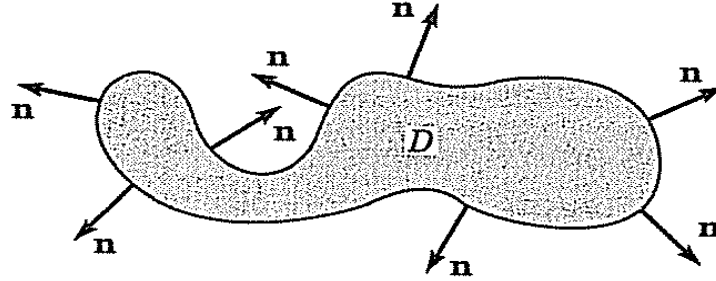


Figure 1

2.5 Second-Order Equations

In general form of PDE for second order linear in two variables x, y with six constant coefficients:

$$a_{11}u_{xx} + 2a_{12}u_{xy} + a_{22}u_{yy} + a_1u_x + a_2u_y + a_0u = 0 \quad (2.11)$$

where $a_{11}, a_{12}, a_{22}, a_1, a_2$ and a_0 are arbitrary constants.

The classification of the second order partial differential equation (2.11) into the following:

i) Elliptic: If $a_{12}^2 < a_{11}a_{22}$, it is reducible to

$$u_{xx} + u_{yy} = 0$$

example of elliptic is Laplace equation in a two dimensional.

ii) Hyperbolic: If $a_{12}^2 > a_{11}a_{22}$, it is

$$u_{xx} - u_{yy} = 0 .$$

example of hyperbolic is wave equation.

iii) Parabolic: If $a_{12}^2 = a_{11}a_{22}$, it is reducible to

$$u_y = ku_{xx}$$

the heat transfer equation is the example of this case.

2.6 Taylor's Theorem

Suppose f and its derivatives first, second and so on like $f', f'', f''', \dots, f^{(n)}$ exist and continuous in $a \leq x \leq b$ and let that $f^{(n+1)}$ exists in $a < x < b$. Then for c in $[a, b]$,

$$f(x) = P_n(x) + R_n(x)$$

where the remainder $R_n(x)$ may be found by any of the following three ways. For any n there exists ξ such that

$$R_n(x) = \frac{1}{(n+1)!} f^{(n+1)}(\xi)(x-c)^{n+1} \quad (\text{Lagrange theorem}) \quad (2.12)$$

(ξ is between c and x .)

For any n there exists ξ such that

$$R_n(x) = \frac{1}{(n)!} f^{(n+1)}(\xi)(x-\xi)^n(x-c) \quad (\text{Cauchy form}) \quad (2.13)$$

$$R_n(x) = \frac{1}{(n)!} \int_c^x (x-t)^n f^{(n+1)}(t) dt \quad (\text{Integral form}) \quad (2.14)$$

If all the derivatives of f exist, with no remainder, one can use the following formula:

$$f(x) = \sum_{n=0}^{\infty} \frac{1}{(n)!} f^{(n)}(c)(x-c)^n \quad (2.15)$$

Eq. (2.15) is Taylor series, if $c = 0$, it is Maclaurin series or expansion.

2.7 The Cauchy Problem

The Schwartz formula the linear differential equation of m th-order for a function $u(x) = u(x_1, \dots, x_n)$ converts to

$$Lu = \sum_{|\alpha| \leq m} A_\alpha(x) D^\alpha u = B(x). \quad (2.16)$$

The same formula defines the general m th-order system of N differential equation in N if we interpret u and B as column vectors with N components and the A_α as square matrices $N \times N$. The general form of m th-order quasi-linear equation is written as

$$Lu = \sum_{|\alpha| < m} A_\alpha D^\alpha u + C = 0, \quad (2.17)$$

where A & C are functions of the independent variable x_k and of the derivatives $D^\beta u$ of the u of orders $|\beta| \leq m - 1$. Farther general nonlinear systems or nonlinear equations

$$F(x, D^\alpha u) = 0 \quad (2.18)$$

can be converted formally to quasi-linear ones using a first-order differential operator to (2.18). On the other hand, a m th-order quasi-linear system (2.17) can be converted to a first-order one, by giving all its derivatives $D^\beta u$ with $|\beta| \leq m - 1$ as new dependent variables, and utilizing of suitable compatibility conditions for the $D^\beta u$.

The Cauchy problem contains finding a solution of u in Eqs. (2.17) or (2.16) having prescribed Cauchy data on hypersurface $S \subset R^n$ presented by

$$\phi(x_1, \dots, x_n) = 0 \quad (2.19)$$

ϕ shall have m continuous derivatives and the surface should be regular in the sense that

$$D_\phi(x_1, \dots, x_n) \neq 0 \quad (2.20)$$

2.8 Real Analytic Functions and the Cauchy-Kovalevsky Theorem

This theorem quite generally asserts the local existence of solutions of the non-characteristic Cauchy problem, provided the equation and data are analytic. Hence we need to consider the class of analytic functions, which plays a significant role in many questions concerning partial differential equations. Global behavior of analytic functions can only be understood when we consider such functions for complex arguments, or as solutions of systems of Cauchy-Riemann equations. However, for most purposes in differential equations it is sufficient to conceive the restrictions of analytic functions to real arguments, the so-called real analytic functions. These form a curious subclass C^ω of C^∞ that can either be characterized by the property of local representability by power series, or by the way their partial derivatives increase with increasing order.

3. Residual Power Series Method (RPSM):

In this section, we give a brief description and some properties of the standard RPSM, which will be used in the remainder of this work, in order to find out series solution for the one dimensional Burgers' and coupled Burger's equations (CBEs) with initial conditions.

We conceive the system of initial value problem

$$\Phi_i^{(n)}(\sigma) = H_i(\sigma, \Phi_i(\sigma), \Phi_i'(\sigma), \dots, \Phi_i^{(n-1)}(\sigma)), i = 1, 2, \dots, r \quad (3.1)$$

subject to the initial conditions

$$\Phi_i(\sigma_0) = \Phi_{i,0}, \Phi_i'(\sigma_0) = \Phi_{i,1}, \dots, \Phi_i^{(n-1)}(\sigma_0) = \Phi_{i,(n-1)} \quad (3.2)$$

where $H_i: (\sigma_0 - \varepsilon, \sigma_0 + \varepsilon) \times R^n \rightarrow R^n$ is a nonlinear analytic function, and σ is the independent variable, $\Phi_i(\sigma), \Phi_i'(\sigma), \dots, \Phi_i^{(n-1)}(\sigma)$ are unknown functions and σ_0, ε are real constants.

Therefore, this solution can be represented as a power series method as:

$$\Phi_i(\sigma) = \sum_{m=0}^{\infty} c_{i,m} (\sigma - \sigma_0)^m \quad (3.3)$$

where the coefficients $c_{i,m}$ are presented as:

$$c_{i,m} = \frac{\Phi_i^{(m)}(\sigma_0)}{m!} = \frac{\Phi_{i,m}}{m!}, m = 0, 1, \dots, n-1. \quad (3.4)$$

According to equations (3.2) - (3.4), the series solution can be written as

$$\begin{aligned} \Phi_i(\sigma) &= \Phi_{i,0} + \Phi_{i,1}(\sigma - \sigma_0) + \frac{\Phi_{i,2}(\sigma - \sigma_0)^2}{2!} + \dots + \frac{\Phi_{i,(n-1)}(\sigma - \sigma_0)^{n-1}}{(n-1)!} \\ &+ \sum_{m=0}^{\infty} c_{i,m} (\sigma - \sigma_0)^m \end{aligned} \quad (3.5)$$

In practice, we approximate solutions by the k^{th} -truncated series

$$\begin{aligned} \Phi_i(\sigma) &= \Phi_{i,0} + \Phi_{i,1}(\sigma - \sigma_0) + \frac{\Phi_{i,2}(\sigma - \sigma_0)^2}{2!} + \dots + \frac{\Phi_{i,(n-1)}(\sigma - \sigma_0)^{n-1}}{(n-1)!} \\ &+ \sum_{m=0}^k c_{i,m} (\sigma - \sigma_0)^m \end{aligned} \quad (3.6)$$

Now, to define the rest of the coefficients $c_{i,m}$ for $m = n, n + 1, \dots, k$ we give k^{th} -residual function by:

$$R_i(\sigma) = \Phi_i^{(n)}(\sigma_0) - H_i(\sigma, \Phi_i(\sigma), \Phi_i'(\sigma), \dots, \Phi_i^{(n-1)}(\sigma)). \quad (3.7)$$

It is clear that $R_i(\sigma) = 0$ for each $\sigma \in (\sigma_0 - \varepsilon, \sigma_0 + \varepsilon)$, this is to verify that these residual functions are differentiable infinitely many times at $\sigma = \sigma_0$. Moreover,

$$\frac{d^m}{dt^m} R_i(\sigma) = 0 \text{ at } \sigma = \sigma_0. \quad (3.8)$$

equations (3.6) and (3.8) for $m = n, n + 1, \dots, k$, generate $k - n + 1$ set of linear and nonlinear equations, respectively. These equations can be solved by symbolic computations software package such as Mathematica and Maple.

4. Applications

4.1 One-dimensional Burgers' equation

Example I Conceive that the one-dimensional Burgers' equation has the form [40]

$$u_t - uu_x - \nu u_{xx} = 0, \quad -1 < x < 1, t \in [0, T] \quad (4.1)$$

subject to the following nonhomogeneous initial condition

$$u(x, 0) = \frac{\sinh \frac{x}{2\nu}}{\cosh \frac{x}{2\nu} + e^{-\frac{1}{4\nu}}}, \quad x \in \Delta \quad (4.2)$$

where $\Delta = \{x \in \mathbb{R} : \mathcal{A} \leq x \leq \mathcal{B}\}$ is the computational domain; and ν is arbitrary constant.

To solve the Eqs. (4.1) with (4.2) by means of RPS method, and the k^{th} -residual functions and the ∞ th-residual functions are given, respectively, by

$$Res^k = (u_k)_t - u(u_k)_x - \nu(u_k)_{xx}, \quad k \geq 1 \quad (4.3)$$

and

$$Res^\infty = \lim_{k \rightarrow \infty} Res^k. \quad (4.4)$$

As in [31-39] it is clear that $Res = 0$ and $\lim_{k \rightarrow \infty} Res^k = Res$ for each $x \in \Delta$ and $t \geq 0$. In fact, this shows that $\left(\frac{\partial^j Res}{\partial t^j} = 0\right)$ when $t = 0$, for each $j = 1, 2, \dots, k$. To define form of $f_1(x)$, we substitute $k = 1$ into both sides of the 1st-residual function that acquired from Eq. (4.3), we get:

$$Res^1 = (u_1)_t - u(u_1)_x - \nu(u_1)_{xx} \quad (4.5)$$

where

$$u_1 = f(x) + f_1(x) t$$

For

$$u_0 = f_0(x) = f(x) = u(x, 0) = \frac{\sinh \frac{x}{2\nu}}{\cosh \frac{x}{2\nu} + e^{-\frac{1}{4\nu}}}$$

From equation (4.5), we conclude that $Res^1 = 0$ and $t = 0$, we get

$$f_1(x) = \frac{e^{\frac{1}{4\nu}} \sinh \frac{x}{2\nu}}{4\nu (1 + e^{\frac{1}{4\nu}} \cosh \frac{x}{2\nu})^2}$$

The first approximation for Eqs. (4.1) and (4.2) as follows:

$$u_1 = \frac{\sinh \frac{x}{2v}}{\cosh \frac{x}{2v} + e^{-\frac{1}{4v}}} + \frac{e^{\frac{1}{4v}} \sinh \frac{x}{2v}}{4v (1 + e^{\frac{1}{4v}} \cosh \frac{x}{2v})^2} t \quad (4.6)$$

Now, in order to get the second approximation, we write $k = 2$ in Eq. (4.3), we get

$$Res^2 = (u_2)_t - u(u_2)_x - v(u_2)_{xx} \quad (4.7)$$

We differentiate Eq. (4.7) with respect to t and substitution of $t = 0$, and $\frac{\partial}{\partial t} Res^2(0) = 0$, we get

$$f_2(x) = -\frac{e^{\frac{1}{4v}} \left(-1 + e^{\frac{1}{4v}} \cosh \frac{x}{2v}\right) \sinh \frac{x}{2v}}{32 v^2 \left(-1 + e^{\frac{1}{4v}} \cosh \frac{x}{2v}\right)^3}$$

Therefore, the second RPS approximate solution are

$$u_2 = \frac{\sinh \frac{x}{2v}}{\cosh \frac{x}{2v} + e^{-\frac{1}{4v}}} + \frac{e^{\frac{1}{4v}} \sinh \frac{x}{2v}}{4v (1 + e^{\frac{1}{4v}} \cosh \frac{x}{2v})^2} t - \frac{e^{\frac{1}{4v}} \left(-1 + e^{\frac{1}{4v}} \cosh \frac{x}{2v}\right) \sinh \frac{x}{2v}}{32 v^2 \left(-1 + e^{\frac{1}{4v}} \cosh \frac{x}{2v}\right)^3} t^2; \quad (4.8)$$

Similarly, we put $k = 3$ in to Eq. (4.3)

$$Res^3 = (u_3)_t - u(u_3)_x - v(u_3)_{xx} \quad (4.9)$$

we write

$$u_3 = f(x) + f_1(x) t + f_2(x) t^2 + f_3(x) t^3;$$

Utilizing $\frac{\partial}{\partial t} Res^3(0) = 0$ and $(t = 0)$ into equation (4.9), we get

$$f_3(x) = \frac{e^{\frac{1}{4v}} \left(2 + e^{\frac{1}{2v}} - 8e^{\frac{1}{4v}} \cosh \frac{x}{2v} + e^{\frac{1}{2v}} \cosh \frac{x}{v}\right) \sinh \frac{x}{2v}}{768 v^3 \left(1 + e^{\frac{1}{4v}} \cosh \frac{x}{2v}\right)^4}$$

Therefore, the third RPS approximate solutions are

$$u_3 = \frac{\sinh \frac{x}{2v}}{\cosh \frac{x}{2v} + e^{-\frac{1}{4v}}} + \frac{e^{\frac{1}{4v}} \sinh \frac{x}{2v}}{4v \left(1 + e^{\frac{1}{4v}} \cosh \frac{x}{2v}\right)^2} t - \frac{e^{\frac{1}{4v}} \left(-1 + e^{\frac{1}{4v}} \cosh \frac{x}{2v}\right) \sinh \frac{x}{2v}}{32 v^2 \left(-1 + e^{\frac{1}{4v}} \cosh \frac{x}{2v}\right)^3} t^2 + \frac{e^{\frac{1}{4v}} \left(2 + e^{\frac{1}{2v}} - 8e^{\frac{1}{4v}} \cosh \frac{x}{2v} + e^{\frac{1}{2v}} \cosh \frac{x}{v}\right) \sinh \frac{x}{2v}}{768 v^3 \left(1 + e^{\frac{1}{4v}} \cosh \frac{x}{2v}\right)^4} t^3; \quad (4.10)$$

To get the form of the $f_4(x)$, we put $k = 4$ in to Eq. (4.3)

$$Res^4 = (u_4)_t - u(u_4)_x - v(u_4)_{xx} \quad (4.11)$$

we write

$$u_4 = f(x) + f_1(x)t + f_2(x)t^2 + f_3(x)t^3 + f_4(x)t^4;$$

Utilizing $\frac{\partial}{\partial t} Res^4(0) = 0$ and $(t = 0)$ into equation (4.11), we get

$$f_4(x) = - \frac{\left(e^{\frac{1}{4v}} \left(-4 - 22e^{\frac{1}{2v}} + e^{\frac{1}{4v}} \left(44 + 3e^{\frac{1}{2v}} \right) \cosh \frac{x}{2v} - 22e^{\frac{1}{2v}} \cosh \frac{x}{v} + e^{\frac{3}{4v}} \cosh \frac{3x}{2v} \right) \sinh \frac{x}{2v} \right)}{24576 v^4 \left(1 + e^{\frac{1}{4v}} \cosh \frac{x}{2v} \right)^5}$$

Thus, the fourth RPS approximate solutions are obtained as:

$$\begin{aligned} u_4 = & \frac{\sinh \frac{x}{2v}}{\cosh \frac{x}{2v} - e^{-\frac{1}{4v}}} + \frac{e^{\frac{1}{4v}} \sinh \frac{x}{2v}}{4v \left(1 + e^{\frac{1}{4v}} \cosh \frac{x}{2v} \right)^2} t - \frac{e^{\frac{1}{4v}} \left(-1 + e^{\frac{1}{4v}} \cosh \frac{x}{2v} \right) \sinh \frac{x}{2v}}{32 v^2 \left(-1 + e^{\frac{1}{4v}} \cosh \frac{x}{2v} \right)^3} t^2 \\ & + \frac{e^{\frac{1}{4v}} \left(2 + e^{\frac{1}{2v}} - 8e^{\frac{1}{4v}} \cosh \frac{x}{2v} + e^{\frac{1}{2v}} \cosh \frac{x}{v} \right) \sinh \frac{x}{2v}}{768 v^3 \left(1 + e^{\frac{1}{4v}} \cosh \frac{x}{2v} \right)^4} t^3 - \\ & \frac{\left(e^{\frac{1}{4v}} \left(-4 - 22e^{\frac{1}{2v}} + e^{\frac{1}{4v}} \left(44 + 3e^{\frac{1}{2v}} \right) \cosh \frac{x}{2v} - 22e^{\frac{1}{2v}} \cosh \frac{x}{v} + e^{\frac{3}{4v}} \cosh \frac{3x}{2v} \right) \sinh \frac{x}{2v} \right)}{24576 v^4 \left(1 + e^{\frac{1}{4v}} \cosh \frac{x}{2v} \right)^5} t^4; \end{aligned} \quad (4.12)$$

and so on, were acquired using the Wolfram Mathematica 9 software package. The exact solution in a closed form is

$$u(x, t) = \frac{\sinh \frac{x}{2v}}{\cosh \frac{x}{2v} + e^{-\frac{t+1}{4v}}}, \quad (4.13)$$

The behavior of the solutions obtained by RPSM is shown for different values of times in Fig. 1 and Fig. 2.

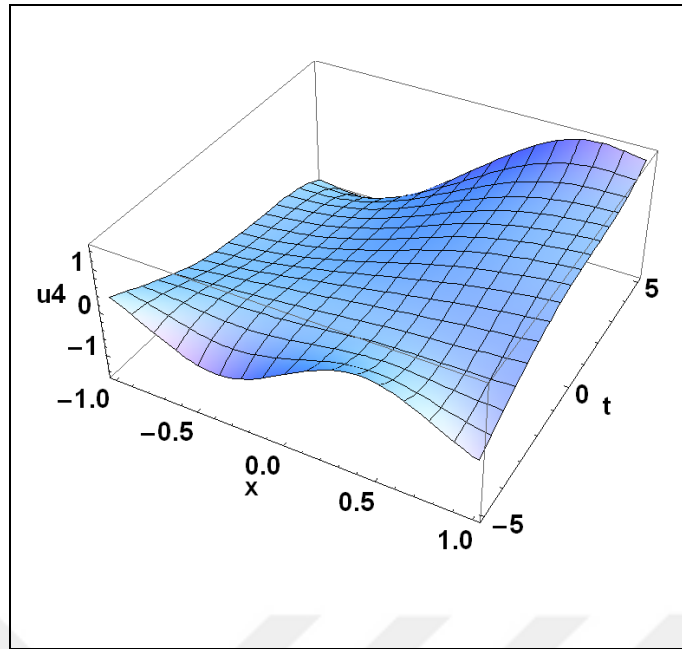


Figure 1.A. 3D surfaces graph of the approximate solution for Eq. (4.1) at $v = 0.3$.

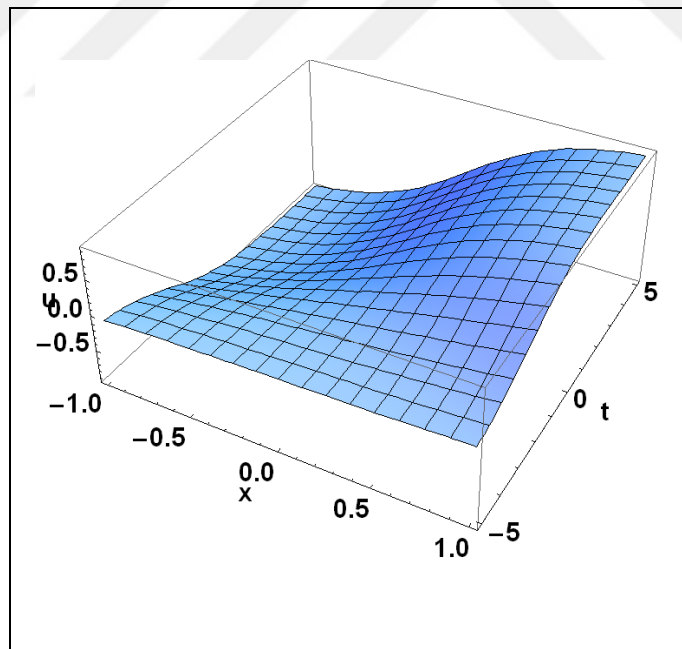


Figure 1.B. 3D surfaces graph of the exact solution for Eq. (4.1) at $v = 0.3$.

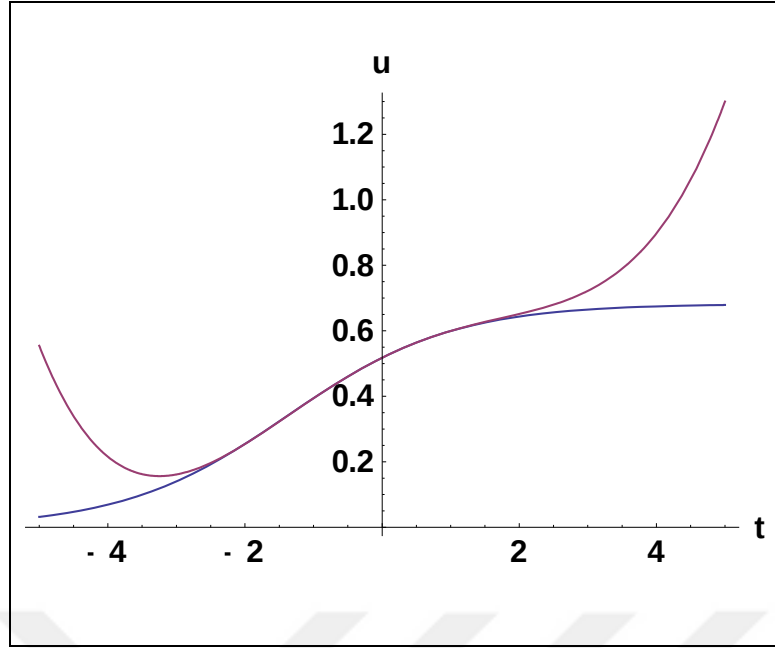


Figure 2. 2D surfaces of the approximate and exact solutions for Eq. (4.1), ($v = 0.3$ and $x = 0.5$).

Table 1. Absolute errors at $v = 0.3$ for Example I.

x / t	0.1	0.2	0.3	0.4	0.5
0.1	3.62210E-11	2.00338E-09	2.14095E-08	1.15372E-07	4.25777E-07
0.2	2.21501E-10	8.63871E-09	7.69222E-08	3.69869E-07	1.26211E-06
0.3	6.41492E-10	2.25181E-08	1.85411E-07	8.38913E-07	2.72660E-06
0.4	1.28128E-09	4.30721E-08	3.41864E-07	1.49893E-06	4.72025E-06
0.5	2.03163E-09	6.67891E-08	5.19576E-07	2.23718E-06	6.95935E-06

Table 1 shows the difference of exact and numerical solutions of the absolute error for $v = 0.3$. We achieved a very good approximation with the exact solution of the equation by using the four terms only of the residual series derived above. It seen that for small values of v , we obtain more proper solution. However, many terms can be calculated in order to achieve a high level of accuracy of the residual power series method with help of Wolfram Mathematica program.

Example II Conceive that the one-dimensional Burgers' equation has the form [41]

$$u_t + uu_x - \nu u_{xx} = 0 \quad (4.14)$$

subject to the following nonhomogeneous initial condition

$$u(x, 0) = 2x, \quad 0 < x < 1 \quad (4.15)$$

To solve the Eqs. (4.14) and (4.15) by means of RPS method, and the k^{th} -residual functions and the ∞^{th} residual functions are given, respectively, by

$$Res^k = (u_k)_t + u(u_k)_x - \nu(u_k)_{xx}, \quad k \geq 1 \quad (4.16)$$

and

$$Res^\infty = \lim_{k \rightarrow \infty} Res^k. \quad (4.17)$$

As in [31-39] it is clear that $Res = 0$ and $\lim_{k \rightarrow \infty} Res^k = Res$ for each $x \in \Delta$ and $t \geq 0$. In fact, this shows that $\left(\frac{\partial^j Res}{\partial t^j} = 0\right)$ when $t = 0$, for each $j = 1, 2, \dots, k$. To find first coefficient, we substitute $k = 1$ into Eq. (4.16), to get:

$$Res^1 = (u_1)_t + u(u_1)_x - \nu(u_1)_{xx} \quad (4.18)$$

where

$$u_1 = f(x) + f_1(x) t$$

For

$$u_0 = f_0(x) = f(x) = u(x, 0) = 2x$$

From equation (4.18), we conclude that $Res^1 = 0$ and $t = 0$, we get

$$f_1(x) = -4x$$

The first RPS approximate solution for Eqs. (4.14) and (4.15) can be written as

$$u_1(x, t) = 2x - 4xt \quad (4.19)$$

In order to get the second RPS approximate solutions, we write $k = 2$ in Eq. (4.16)

$$Res^2 = (u_2)_t + u(u_2)_x - \nu(u_2)_{xx} \quad (4.20)$$

We differentiate Eq. (4.20) with respect to t and substitution $t = 0$, and $\frac{\partial}{\partial t} Res^2(0) = 0$, we get

$$u_2(x, t) = 2x - 4xt + 8xt^2; \quad (4.21)$$

Similarly, we put $k = 3$ in to Eq. (4.16)

$$Res^3 = (u_3)_t + u(u_3)_x - v(u_3)_{xx} \quad (4.22)$$

we write

$$u_3 = f(x) + f_1(x)t + f_2(x)t^2 + f_3(x)t^3;$$

Utilizing $\frac{\partial}{\partial t} Res^3(0) = 0$ and $(t = 0)$ into equation (4.22), we get

$$u_3(x, t) = 2x - 4xt + 8xt^2 - \frac{64}{3}t^3; \quad (4.23)$$

Similarly, to find out the form of the $f_4(x)$, we put $k = 4$ in to Eq. (4.16)

$$Res^4 = (u_4)_t + u(u_4)_x - v(u_4)_{xx} \quad (4.24)$$

we write

$$u_4 = f(x) + f_1(x)t + f_2(x)t^2 + f_3(x)t^3 + f_4(x)t^4;$$

The fact that $\frac{\partial}{\partial t} Res^4(0) = 0$ and $(t = 0)$ into equation (4.24), we get

$$u_4(x, t) = 2x - 4xt + 8xt^2 - 16xt^3 + 32xt^4. \quad (4.25)$$

Thus we have

$$u(x, t) = Res^\infty[u_n(x, t)] = 2x(1 - 2t + 4t^2 - 8t^3 + \dots)$$

$$= \sum_{m=0}^{\infty} (-1)^m 2^{m+1} x t^m = \frac{2x}{1 + 2t}.$$

This is an exact solution.

These results are plotted in Figure 3 and Figure 4.

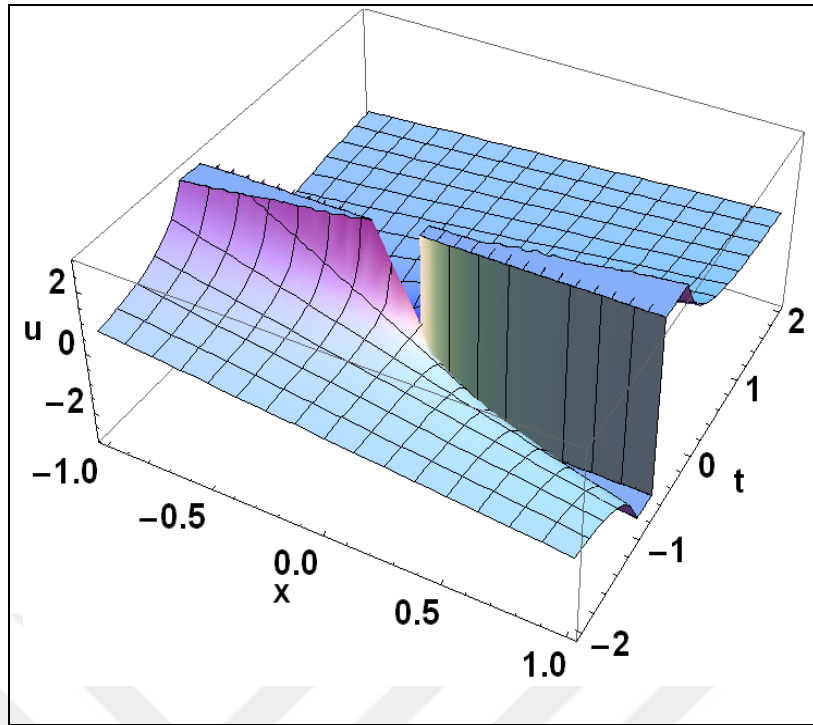


Figure 3. 3D Surfaces graph of the approximate and exact solutions for Eq. (4.14).
($v = 1$).

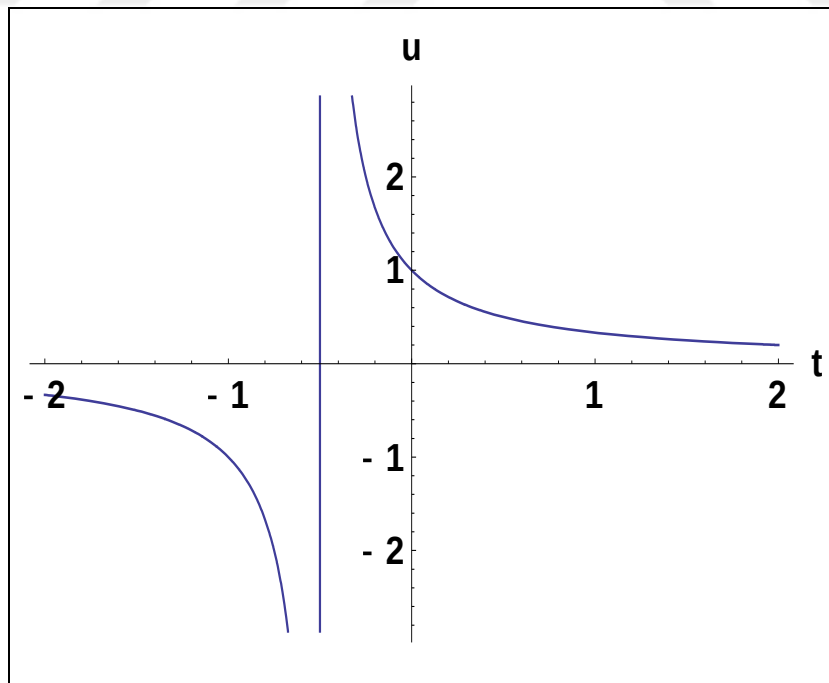


Figure 4. 2D surface of the approximate and exact solutions of Eq. (4.14) at $v = 1$ and $x = 0.5$.

4.2. Two-dimensional coupled Burgers' system

We now conceive that the two-dimensional Burgers' system has the form [42-44]

$$u_t + uu_x + vu_y = \frac{1}{R}(u_{xx} - u_{yy}), \quad (4.26)$$

$$v_t + uv_x + vv_y = \frac{1}{R}(v_{xx} - v_{yy}), \quad (4.27)$$

with the following initial conditions

$$\left. \begin{aligned} u(x, y, 0) &= \frac{3}{4} - \frac{1}{4 \left(1 + e^{\frac{R(-x+y)}{8}} \right)} \\ v(x, y, 0) &= \frac{3}{4} + \frac{1}{4 \left(1 + e^{\frac{R(-x+y)}{8}} \right)} \end{aligned} \right\}, \quad (x, y) \in \Delta_1 \quad (4.28)$$

where $\Delta_1 = \{(x, y) \in \mathbb{R} : \mathcal{A} \leq x \leq \mathcal{B}, \mathcal{C} \leq y \leq \mathcal{D}\}$; and R is the Reynolds number; and v is arbitrary constant.

Before implementing the RPS method for obtaining form of the coefficients $f_n(x, y)$, $n = 1, 2, 3, \dots, k$ in the series expansion, we must determine the residual function concept for Eqs. (4.26) and (4.27) as

$$Res = u_t + uu_x + vu_y - \frac{1}{R}(u_{xx} - u_{yy}) \quad (4.29)$$

$$Res = v_t + uv_x + vv_y - \frac{1}{R}(v_{xx} - v_{yy}) \quad (4.30)$$

Substitution of k^{th} -truncated series u_k and v_k into Eqs. (4.29) and (4.30) gives:

$$Res^k = (u_k)_t + u(u_k)_x + v(u_k)_y - \frac{1}{R}((u_k)_{xx} - (u_k)_{yy}) \quad (4.31)$$

$$Res^k = (v_k)_t + u(v_k)_x + v(v_k)_y - \frac{1}{R}((v_k)_{xx} - (v_k)_{yy}) \quad (4.32)$$

$$k = 1, 2, 3, \dots$$

As in [31-39] it is obvious that $Res = 0$ and $\lim_{k \rightarrow \infty} Res^k = Res$ for each $(x, y) \in \Omega_1$ and $t \geq 0$. In fact, this shows that $\left(\frac{\partial^j Res}{\partial t^j} = 0 \right)$ when $t = 0$, for each $j = 1, 2, \dots, k$. To define form of the $f_1(x, y)$ and $g_1(x, y)$, we should substitute $k = 1$ into both sides of the 1st-residual function that acquired from Eqs. (4.31) and (4.32), to get the following result:

$$Res^1 = (u_1)_t + u(u_1)_x + v(u_1)_y - \frac{1}{R}((u_1)_{xx} - (u_1)_{yy}) \quad (4.33)$$

$$Res^1 = (v_1)_t + u(v_1)_x + v(v_1)_y - \frac{1}{R}((v_1)_{xx} - (v_1)_{yy}) \quad (4.34)$$

where

$$u_1 = f(x, y) + f_1(x, y) t$$

$$v_1 = g(x, y) + g_1(x, y) t$$

For

$$u_0 = f_0(x, y) = f(x, y) = u(x, y, 0) = \frac{3}{4} - \frac{1}{4 \left(1 + e^{\frac{R(-x+y)}{8}} \right)}$$

$$v_0 = g_0(x, y) = g(x, y) = v(x, y, 0) = \frac{3}{4} + \frac{1}{4 \left(1 + e^{\frac{R(-x+y)}{8}} \right)}$$

And using the fact that $Res^1 = 0$, $(t = 0)$ into equation (4.33) and (4.34), and thus

$$f_1(x, y) = - \frac{e^{\frac{R(-x+y)}{8}} R}{128 \left(1 + e^{\frac{R(-x+y)}{8}} \right)^2}$$

$$g_1(x, y) = \frac{e^{\frac{R(-x+y)}{8}} R}{128 \left(1 + e^{\frac{R(-x+y)}{8}} \right)^2}$$

Thus, using 1st-truncated series expansion, the first approximate solution for system of CBEs (4.26) and (4.27) with (4.28) can be written as

$$u_1 = \frac{3}{4} - \frac{1}{4 \left(1 + e^{\frac{R(-x+y)}{8}} \right)} - \frac{e^{\frac{R(-x+y)}{8}} R t}{128 \left(1 + e^{\frac{R(-x+y)}{8}} \right)^2} \quad (4.35)$$

$$v_1 = \frac{3}{4} + \frac{1}{4 \left(1 + e^{\frac{R(-x+y)}{8}} \right)} + \frac{e^{\frac{R(-x+y)}{8}} R t}{128 \left(1 + e^{\frac{R(-x+y)}{8}} \right)^2} \quad (4.36)$$

To get the 2nd RPS approximate solution, we take $k = 2$ and $u_2 = \sum_{m=0}^2 f_m(x) t^m$ and $v_2 = \sum_{m=0}^2 g_m(x) t^m$. We differentiate both sides of Eqs. (4.31) and (4.32) with respect to t and substitute $t = 0$

$$\frac{\partial}{\partial t} Res^2 = \frac{\partial}{\partial t} \left[(u_2)_t + u(u_2)_x + v(u_2)_y - \frac{1}{R}((u_2)_{xx} - (u_2)_{yy}) \right] \quad (4.37)$$

$$\frac{\partial}{\partial t} Res^2 = \frac{\partial}{\partial t} \left[(v_2)_t + u(v_2)_x + v(v_2)_y - \frac{1}{R}((v_2)_{xx} - (v_2)_{yy}) \right] \quad (4.38)$$

Using the fact that $\frac{\partial}{\partial t} Res^2(0) = 0$, and solving the resultant algebraic equation for $f_2(x, y)$ in Eq. (4.37) and $g_2(x, y)$ into Eq. (4.38), we can easily obtain

$$f_2(x, y) = -\frac{e^{\frac{R(-x+y)}{8}} \left(-1 + e^{\frac{R(-x+y)}{8}}\right) R^2}{8192 \left(1 + e^{\frac{R(-x+y)}{8}}\right)^3}$$

$$g_2(x, y) = \frac{e^{\frac{R(-x+y)}{8}} \left(-1 + e^{\frac{R(-x+y)}{8}}\right) R^2}{8192 \left(1 + e^{\frac{R(-x+y)}{8}}\right)^3}$$

Hence, using 2nd-truncated series expansion; the second RPS approximate solution for system of CBEs (4.26) and (4.27) with (4.28) can be written as

$$u_2 = \frac{3}{4} - \frac{1}{4 \left(1 + e^{\frac{R(-x+y)}{8}}\right)} - \frac{e^{\frac{R(-x+y)}{8}} R t}{128 \left(1 + e^{\frac{R(-x+y)}{8}}\right)^2} - \frac{e^{\frac{R(-x+y)}{8}} \left(-1 + e^{\frac{R(-x+y)}{8}}\right) R^2 t^2}{8192 \left(1 + e^{\frac{R(-x+y)}{8}}\right)^3}, \quad (4.39)$$

$$v_2 = \frac{3}{4} - \frac{1}{4 \left(1 + e^{\frac{R(-x+y)}{8}}\right)} + \frac{e^{\frac{R(-x+y)}{8}} R t}{128 \left(1 + e^{\frac{R(-x+y)}{8}}\right)^2} + \frac{e^{\frac{R(-x+y)}{8}} \left(-1 + e^{\frac{R(-x+y)}{8}}\right) R^2 t^2}{8192 \left(1 + e^{\frac{R(-x+y)}{8}}\right)^3}, \quad (4.40)$$

Similarly, we write

$$u_3 = f(x, y) + f_1(x, y) t + f_2(x, y) t^2 + f_3(x, y) t^3$$

$$v_3 = g(x, y) + g_1(x, y) t + g_2(x, y) t^2 + g_3(x, y) t^3$$

in Res^3 .

$\frac{\partial}{\partial t} Res^3(0) = 0$ ($t = 0$) and thus

$$f_3(x, y) = -\frac{e^{\frac{R(-x+y)}{8}} \left(1 - 4e^{\frac{R(-x+y)}{8}} + 1 + e^{\frac{R(-x+y)}{4}}\right) R^3}{786432 \left(1 + e^{\frac{R(-x+y)}{8}}\right)^4}$$

$$g_3(x, y) = \frac{e^{\frac{R(-x+y)}{8}} \left(1 - 4e^{\frac{R(-x+y)}{8}} + 1 + e^{\frac{R(-x+y)}{4}}\right) R^3}{786432 \left(1 + e^{\frac{R(-x+y)}{8}}\right)^4}$$

Therefore, using 3rd-truncated series expansion; the third approximate solution for system of CBEs (4.26) and (4.27) with (4.28) can be written as

$$u_3 = \frac{3}{4} - \frac{1}{4 \left(1 + e^{\frac{R(-x+y)}{8}}\right)} - \frac{e^{\frac{R(-x+y)}{8}} R t}{128 \left(1 + e^{\frac{R(-x+y)}{8}}\right)^2} - \frac{e^{\frac{R(-x+y)}{8}} \left(-1 + e^{\frac{R(-x+y)}{8}}\right) R^2 t^2}{8192 \left(1 + e^{\frac{R(-x+y)}{8}}\right)^3} - \frac{e^{\frac{R(-x+y)}{8}} \left(1 - 4e^{\frac{R(-x+y)}{8}} + 1 + e^{\frac{R(-x+y)}{4}}\right) R^3 t^3}{786432 \left(1 + e^{\frac{R(-x+y)}{8}}\right)^4} \quad (4.41)$$

$$v_3 = \frac{3}{4} + \frac{1}{4 \left(1 + e^{\frac{R(-x+y)}{8}}\right)} + \frac{e^{\frac{R(-x+y)}{8}} R t}{128 \left(1 + e^{\frac{R(-x+y)}{8}}\right)^2} + \frac{e^{\frac{R(-x+y)}{8}} \left(-1 + e^{\frac{R(-x+y)}{8}}\right) R^2 t^2}{8192 \left(1 + e^{\frac{R(-x+y)}{8}}\right)^3} + \frac{e^{\frac{R(-x+y)}{8}} \left(1 - 4e^{\frac{R(-x+y)}{8}} + 1 + e^{\frac{R(-x+y)}{4}}\right) R^3 t^3}{786432 \left(1 + e^{\frac{R(-x+y)}{8}}\right)^4} \quad (4.42)$$

By the same method, the process can be repeated for finding coefficients of RPS approximate solutions for system of coupled burger's equations and initial conditions are acquired.

Table 2 demonstrates the results of the RPSM in comparison with discrete Adomian decomposition method (DADM) [43] and the implicit finite difference method (IFDM) [45]. A good approximate solution with the actual solution of the equations is obtained by utilizing only four terms of the residual power series. The absolute errors show that the numerical approximate solution of RPSM better than the DADM and the IFDM at $t = 0.01$. It gives similar results as those of DADM and IFDM at $t = 0.5$. The experimental results demonstrate that the RPSM may serve as an alternative to the solution of nonlinear problems in engineering and mathematical physics.

Table 2. Comparison of absolute errors for $u(x, y, t)$ at $R = 100$ and different time.

		t=0.01			t=0.5		
(x,y)	DADM	IFDM	RPSM	DADM	IFDM	RPSM	
(0.1,0.1)	5.91368E-5	7.24132E-5	1.55205E-11	2.77661E-4	5.13431E-4	3.88991E-6	
(0.5,0.1)	4.84030E-6	2.42869E-5	5.96291E-11	4.52018E-4	8.85712E-4	2.89136E-6	
(0.9,0.1)	3.41000E-8	8.39751E-6	4.47864E-13	3.37443E-6	6.53372E-5	2.12339E-8	
(0.3,0.3)	5.91368E-5	8.25331E-5	1.55205E-11	2.77664E-4	7.31601E-4	3.88991E-6	
(0.7,0.3)	4.84403E-6	3.43163E-5	5.96291E-11	4.52081E-4	6.27245E-4	2.89136E-6	
(0.1,0.5)	1.64290E-6	5.62014E-5	6.02943E-11	2.86553E-4	4.01942E-4	4.99468E-6	
(0.5,0.5)	5.91368E-5	7.32522E-5	1.55205E-11	2.77664E-4	3.46823E-4	3.88991E-6	

Table 3

Comparison between RPS approximate solutions u_3 and exact solution u when $y = 1, R = 100$. For eq. (4.26).

t	x	Numerical solution	Exact solution	Absolute error
0.1	0.1	0.749995556739048	0.749995555362034	1.37701E-9
	0.2	0.749984492179583	0.749984487375994	4.80359E-9
	0.3	0.749945880721293	0.749945863987526	1.67338E-8
	0.4	0.749811206604694	0.749811148591817	5.80129E-8
	0.5	0.749342279240630	0.749342081506278	1.97734E-7
0.2	0.1	0.749993948488277	0.749993924939814	2.35485E-8
	0.2	0.749978879382691	0.749978797239569	8.21431E-8
	0.3	0.749926296831746	0.749926010721621	2.8611E-7
	0.4	0.749742933605494	0.749741942240811	9.91365E-7
	0.5	0.749104972060236	0.749101599354644	3.37271E-6
0.3	0.1	0.749991824238092	0.749991696451263	1.27787E-7
	0.2	0.749971465893956	0.749971020164203	4.4573E-7
	0.3	0.749900431844389	0.749898879625539	1.55222E-6
	0.4	0.749652785280290	0.749647410375845	5.3749E-6
	0.5	0.748791892578426	0.748773648573569	1.8244E-5
0.4	0.1	0.749989084761012	0.749988650532824	4.34228E-7
	0.2	0.749961905465185	0.749960390945224	1.51452E-6
	0.3	0.749867078327028	0.749861805340769	5.27299E-6
	0.4	0.749536560566201	0.749518316334168	1.82442E-5
	0.5	0.748388538580637	0.748326787268928	6.17513E-5
0.5	0.1	0.749985630829558	0.749984487375994	1.14345E-6
	0.2	0.749949851848187	0.749945863987526	3.98786E-6
	0.3	0.749825028847467	0.749811148591817	1.38803E-5
	0.4	0.749390058400348	0.749342081506278	4.79769E-5
	0.5	0.747880407852308	0.747718590652704	0.000161817

Table 4

Comparison between RPS approximate solutions v_3 and exact solution v when $y = 1, R = 100$. For eq. (4.27).

t	x	Numerical solution	Exact solution	Absolute error
0.1	0.1	0.750004443260951	0.750004444637965	1.37701E-9
	0.2	0.750015507820416	0.750015512624005	4.80359E-9
	0.3	0.750054119278706	0.750054136012473	1.67338E-8
	0.4	0.750188793395305	0.750188851408182	5.80129E-8
	0.5	0.750657720759369	0.750657918493721	1.97734E-7
0.2	0.1	0.750006051511722	0.750006075060185	2.35485E-8
	0.2	0.750021120617308	0.750021202760430	8.21431E-8
	0.3	0.750073703168253	0.750073989278378	2.8611E-7
	0.4	0.750257066394505	0.750258057759188	9.91365E-7
	0.5	0.750895027939763	0.750898400645355	3.37271E-6
0.3	0.1	0.750008175761907	0.750008303548736	1.27787E-7
	0.2	0.750028534106043	0.750028979835796	4.4573E-7
	0.3	0.750099568155610	0.750101120374460	1.55222E-6
	0.4	0.750347214719709	0.750352589624154	5.3749E-6
	0.5	0.751208107421574	0.751226351426430	1.8244E-5
0.4	0.1	0.750010915238987	0.750011349467175	4.34228E-7
	0.2	0.750038094534814	0.750039609054775	1.51452E-6
	0.3	0.750132921672971	0.750138194659230	5.27299E-6
	0.4	0.750463439433798	0.750481683665831	1.82442E-5
	0.5	0.751611461419363	0.751673212731071	6.17513E-5
0.5	0.1	0.750014369170441	0.750015512624005	1.14345E-6
	0.2	0.750050148151812	0.750054136012473	3.98786E-6
	0.3	0.750174971152532	0.750188851408182	1.38803E-5
	0.4	0.750609941599651	0.750657918493721	4.79769E-5
	0.5	0.752119592147691	0.752281409347295	0.000161817

The numerical results of Application 4.2 are shown in Table 3 and Table 4 and Figures 5, 6 and 7 when $y = 1, R = 100$. Table shows RPS approximation u_3 and exact solution u of the tow dimensional burger's equations at different times $t = 0.1, 0.2, 0.3, 0.4, 0.5$. The Figures. 5 and 6 compares the numerical results with the exact one respectively in 3D form. From numerical results of this application, we conclude that the obtained results are quite agreed with the exact one.

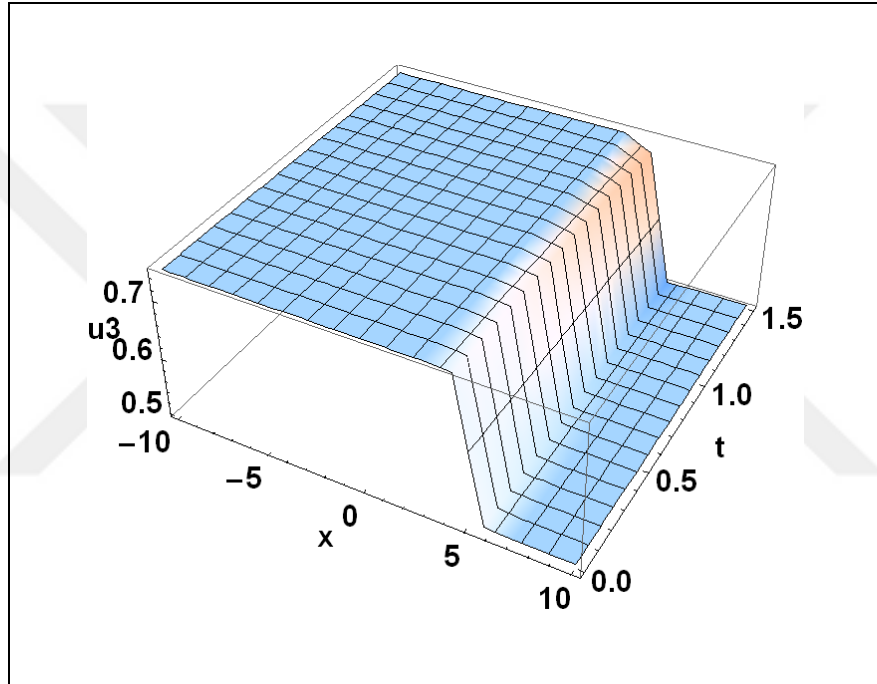


Figure 5. 3D surfaces of the residual power series approximate and exact solution for different time with $R = 50, y = 5$, for Eq. (4.26).

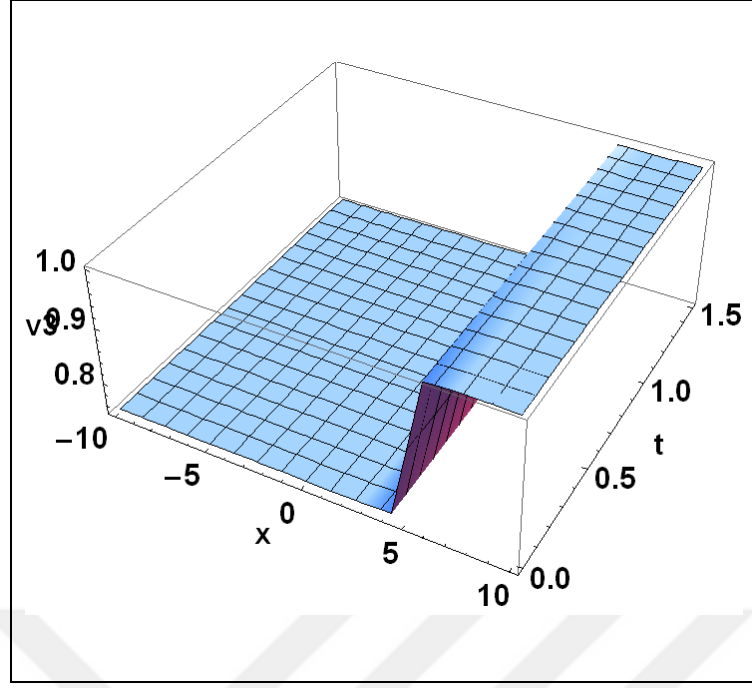


Figure 6. 3D surfaces of the residual power series approximate and exact solution for different time with $R = 50, y = 5$, for Eq. (4.27).

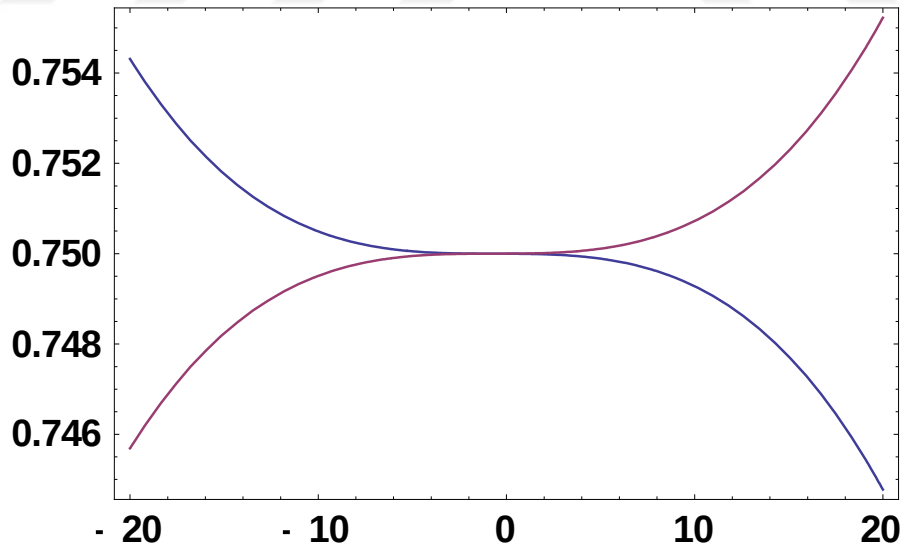


Figure 7. 2D surfaces graph of $v_3(x, y, t)$ and $u_3(x, y, t)$ with fixed values $R = 50, y = 5, x = 3$ for Eqs. (4.26) and (4.27) (u blue, v red).

4.3. Inhomogeneous one-dimensional Burgers' equation

We consider the following coupled Burger's equation has the form

$$u_t - u_{xx} - uu_x = f(x, t) \quad (4.43)$$

where $0 \leq t \leq 1$, $0 \leq x \leq 1$ and $f(x, t) = x^2 - x - 2t - t^2x(x - 1)(2x - 1)$.

subject to the initial condition

$$u(x, 0) = 0. \quad (4.44)$$

To solve the Eqs. (4.43) and (4.44) by means of RPS method, and the k^{th} -residual functions and the ∞^{th} residual functions are given, respectively, by

$$Res^k = (u_k)_t - (u_k)_{xx} - u(u_k)_x - f(x, t) \quad , \quad k = 1, 2, 3, \dots \quad (4.45)$$

and

$$Res^\infty = \lim_{k \rightarrow \infty} Res^k. \quad (4.46)$$

As in Abu Arqub and colleagues [31-39] it is clear that $Res = 0$ and $\lim_{k \rightarrow \infty} Res^k = Res$ for each $x \in \Omega$ and $t \geq 0$. In fact, this shows that $\left(\frac{\partial^j Res}{\partial t^j} = 0\right)$ when $t = 0$, for each $j = 1, 2, \dots, k$. On the other aspect as well, to define form of the first coefficient $f_1(x)$, we substitute $k = 1$ into both sides of the 1st-residual function that obtained from Eq. (4.45), to obtain:

$$Res^1 = (u_1)_t - (u_1)_{xx} - u(u_1)_x - f(x, t) \quad (4.47)$$

where

$$u_1 = f(x) + f_1(x) t$$

For

$$u_0 = f_0(x) = f(x) = u(x, 0) = 0$$

From equation (4.47), we conclude that $Res^1 = 0$ and $t = 0$, we get

$$f_1(x) = (-1 + x)x$$

The first RPS approximate solution for Eqs. (4.43) and (4.44) can be given as:

$$u_1 = t(-1 + x)x \quad (4.48)$$

(4.43) and (4.44) are obtained as:

$$Res^2 = (u_2)_t - (u_2)_{xx} - u(u_2)_x - f(x, t) \quad (4.49)$$

where

$$u_2 = f(x) + f_1(x) t + f_2(x)t^2$$

In order to get the values of the coefficient $f_2(x)$ in Eq. (4.49), we take $k = 2$ through Eq. (4.45) and utilizing $\frac{\partial}{\partial t} Res^2(0) = 0$ to obtain the following results:

$$f_2(x) = 0$$

Hence, the second RPS approximate solution of Eqs. (4.43) and (4.44), can be expressed as

$$u_2 = t (-1 + x)x ; \tag{4.50}$$

Similarly, we put $k = 3$ in to Eq. (4.45)

$$Res^3 = (u_3)_t - (u_3)_{xx} - u(u_3)_x - f(x, t) \tag{4.51}$$

we write

$$u_3 = f(x) + f_1(x) t + f_2(x)t^2 + f_3(x)t^3;$$

Utilizing $\frac{\partial}{\partial t} Res^3(0) = 0$ and $(t = 0)$ into equation (4.51), we get

$$f_3(x) = 0 ;$$

Therefore, the third RPS approximate solutions are obtained as:

$$u_3 = t (-1 + x)x ; \tag{4.52}$$

Thus, we get the exact solution with three iterations. These results are plotted in Figures 8 and 9.

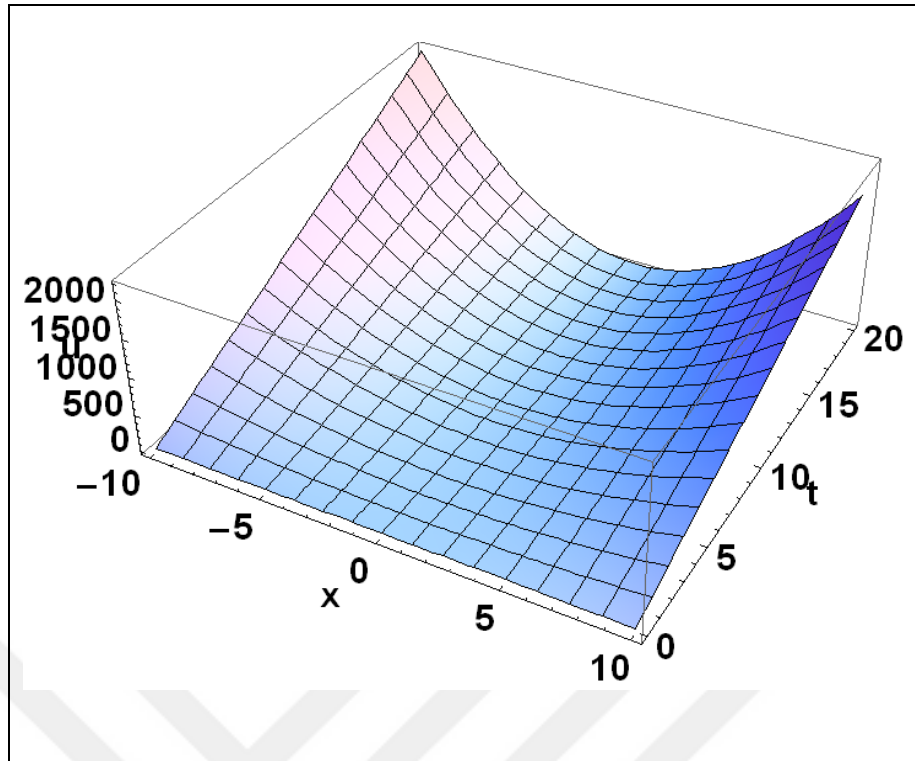


Figure 8. 3D surfaces of the RPSM and exact solution for Eq. (4.43).

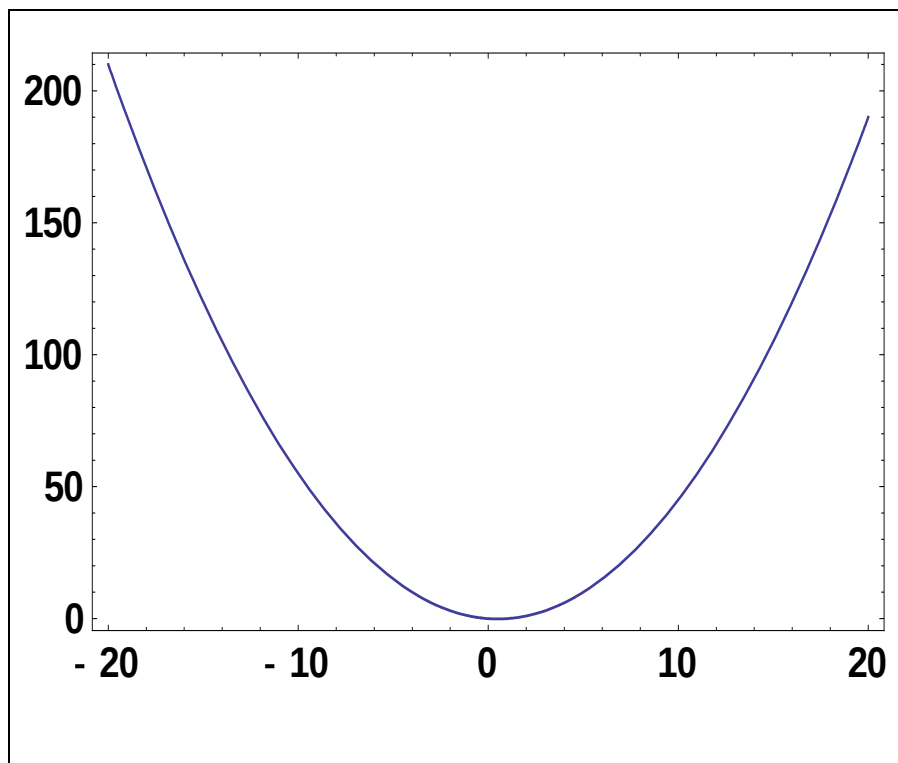


Figure 9. 2D surfaces of the RPSM and exact solution for Eq. (4.43).

5. Conclusion and Discussion

In this thesis, the fundamental point of this study has been to present an effective technique for the solution of the one-dimensional homogeneous and in homogeneous Burgers' equations and coupled Burgers' equation with initial conditions. The main purpose has been acquired by giving the RPSM to investigate these problems. The residual power series method is efficient and significant method in obtaining approximate solutions for nonlinear Burgers' and coupled Burgers' equations with initial conditions. Also, the method can be applicable for one, two and multi-dimensional problems arise in biological, physical and chemical phenomena.

The RPS technique is useful for getting a closed form and the explicit solution and numerical approximate solutions of linear or nonlinear differential equations and fractional PDEs, and it is also quite straightforward to write computer codes. This technique has been implemented to acquire formal solution to a wide class of stochastic and deterministic problems in science and engineering involving algebraic, differential, integral differential, integral and PDEs.

Throughout this work, all numerical computations and all graphs are found by Mathematica 9 software package.

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