

Ambarzumyan Type Theorem for a Quadratic Sturm-Liouville Operator

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Abstract

We consider a quadratic Sturm-Liouville problem. In this paper, some uniqueness theorems are extended to the case in which the governing second-order ordinary differential equation contains both $q(x)$ and $p(x)$ instead of only $q(x)$. It is shown that if the spectrum is the same as the spectrum belonging to the zero potential, then the functions $q(x)$ and $p(x)$ are zero.

Keywords: Quadratic Sturm-Liouville equation, Spectrum, Ambarzumyan's theorem. **MSC 2000:** 34B24, 34A55.

Bir Quadratik Sturm-Liouville Operatorü için Ambarzumyan Tipi Teorem

Özet

Bu çalışmada bir kuadratik Sturm-Liouville problemi ele alındı. Bazı teklik teoremleri, içerisinde sadece $q(x)$ değil hem $p(x)$ hemde $q(x)$ bulunduran ikinci mertebeden adi diferensiyel denklem olması durumuna genişletildi. Spektrum eğer sıfır potansiyeline ait olan spektrum ile aynı ise, $p(x)$ ve $q(x)$ fonksiyonlarının sıfır olduğu gösterildi.

Anahtar Kelimeler: Quadratik Sturm-Liouville denklemi, Spektrum, Ambarzumyan teoremi. **MSC 2000:** 34B24, 34A55.

1. Introduction

Initially, consider the following eigenvalue problem

$$\left. \begin{aligned} -y'' + q(x)y &= \lambda y, & \text{on } [0, \pi] \\ y(0)\cos\theta + y'(0)\sin\theta &= 0 \\ y(\pi)\cos\chi + y'(\pi)\sin\chi &= 0 \end{aligned} \right\} \quad (1.1)$$

where $q(x) \in L^2[0, \pi]$, and $\theta, \chi \in [0, \pi]$. This eigenvalue problem arises in many fields such as mechanics, physics, electronics, geophysics, meteorology and other branches of sciences and there is a lot of literature on solving this problem [3], [10], [14], [16], [18], [19]. Ambarzumyan's paper can be viewed as first and vital reference in the history of inverse spectral problems

associated with Sturm-Liouville operators [1]. In 1929, he showed that for the Neumann boundary

conditions $\left(\theta = \chi = \frac{\pi}{2}\right)$, if the spectrum (collection of the eigenvalues) in (1.1) is $\{\lambda_n = n^2 : n = 0, 1, 2, \dots\}$, then the potential function $q(x)$ is zero almost everywhere on $[0, \pi]$. Ambarzumyan's theorem was extended to the second order differential systems of two dimensions in [5], to Sturm-Liouville differential systems of any dimension in [6], to the Sturm-Liouville equation (which is concerned only with Neumann boundary conditions) with general boundary conditions by imposing an additional condition on the potential function [7], and to the multi-dimensional Dirac operator in [21]. In addition, some different results of

Ambarzumyan's theorem have been obtained in [4], [11], [12], [20], [22], [23].

In this study, by extending the results of classical Sturm-Liouville problem, we show that an explicit formula of eigenvalues can determine two functions in the quadratic pencil of Sturm-Liouville operator with Neumann conditions.

Before giving the main results, we mention some physical properties of the quadratic equation. The problem of describing the interactions between colliding particles is of fundamental interest in physics. It is interesting in collisions of two spinless particles, and it is supposed that the s -wave scattering matrix and the a s -wave binding energies are exactly known from collision experiments. For a radial static potential $V(E, x)$ and a s -wave, the Schrödinger equation is written as

$$y'' + [E - V(E, x)]y = 0,$$

where

$$V(E, x) = 2\sqrt{E}p(x) + q(x).$$

We note that with the additional condition

$q(x) = -p^2(x)$, above equation reduces to the Klein-Gordon s -wave equation for a particle of zero mass and energy $\sqrt{E}$ [15].

Consider the boundary-value problem generated by the quadratic (in the eigenvalue λ) differential equation

$$-y'' + [q(x) + 2\lambda p(x)]y = \lambda^2 y, \quad x \in (0, \pi) \quad (1.2)$$

with the homogeneous Neumann boundary conditions

$$y'(0) = 0, \quad (1.3)$$

$$y'(\pi) = 0, \quad (1.4)$$

In equation (1.2), λ is a spectral parameter and $q(x) \in W_2^1[0, \pi]$, $p(x) \in W_2^2[0, \pi]$. This problem is called the quadratic pencil of the Schrodinger operator. If $p(x) = 0$ the classical Sturm-Liouville operator is obtained. Some versions of the eigenvalue problem (1.2)-(1.4) were studied extensively in [2], [8], [9], [13], [17].

We define

$$\Delta(\lambda) = \varphi'(\pi, \lambda) \quad (1.5)$$

which is called the characteristic function. In the Sturm-Liouville theory, it is well-known that λ is an eigenvalue of the problem (1.2)-(1.4) if and

only if $\Delta(\lambda) = 0$.

Theorem 1. ([8]) Let $q(x) \in W_2^1[0, \pi]$, $p(x) \in W_2^2[0, \pi]$ and $\varphi(x, \lambda)$ is the solution of (1.2) with initial condition (1.3)

$$\varphi(x, \lambda) = \cos(\lambda x - \alpha(x))$$

$$+ \int_0^x A(x, t) \cos(\lambda t) dt \\ + \int_0^x B(x, t) \sin(\lambda t) dt,$$

(1.6)

where $A(x, t)$ and $B(x, t)$ satisfy the following equations

$$\frac{\partial^2 A(x, t)}{\partial x^2} - 2p(x) \frac{\partial B(x, t)}{\partial t} - q(x)A(x, t) \\ = \frac{\partial^2 A(x, t)}{\partial t^2}$$

$$\frac{\partial^2 B(x, t)}{\partial x^2} + 2p(x) \frac{\partial A(x, t)}{\partial t} - q(x)B(x, t) \\ = \frac{\partial^2 B(x, t)}{\partial t^2}$$

$$A(0, 0) = 0, \quad B(x, 0) = 0, \quad \left. \frac{\partial A(x, t)}{\partial t} \right|_{t=0} = 0,$$

$$q(x) + p^2(x) = 2 \frac{d}{dx} \left[\begin{array}{c} A(x, x) \cos \alpha(x) \\ + \\ B(x, x) \sin \alpha(x) \end{array} \right],$$

$$A(0, 0) = 0, \quad B(x, 0) = 0, \quad \left. \frac{\partial A(x, t)}{\partial t} \right|_{t=0} = 0,$$

$$\alpha(x) = \int_0^x p(t) dt = p(0)x \\ + 2 \int_0^x \left[\begin{array}{c} A(\zeta, \zeta) \sin \alpha(\zeta) \\ - \\ B(\zeta, \zeta) \cos \alpha(\zeta) \end{array} \right] d\zeta. \quad (1.7)$$

Lemma 1. ([8]) (1.2)-(1.4) boundary value problem has a countable set of eigenvalues $\{\lambda_n\}$ given by

$$\lambda_n = n + c_0 + \frac{c_1}{n} + \frac{c_{1,n}}{n}, n \geq 1, \quad (1.8)$$

where

$$\begin{aligned} c_0 &= \frac{1}{\pi} \int_0^\pi p(x) dx, c_1 \\ &= \frac{1}{2\pi} \int_0^\pi [q(x) + p^2(x)] dx, \quad (1.9) \\ \sum_n |c_{1,n}|^2 &< \infty, \end{aligned}$$

2. Main Results

In this section, some uniqueness theorems are given for the equation (1.2) with Neumann boundary conditions. It is shown that an explicit formula of eigenvalues can determine the functions $q(x)$ and $p(x)$ be zero both.

Consider a second quadratic Sturm - Liouville problem

$$-y'' + [\tilde{q}(x) + 2\lambda p(x)]y = \lambda^2 y, x \in (0, \pi) \quad (2.1)$$

$$y'(0) = 0, \quad (2.2)$$

$$y'(\pi) = 0, \quad (2.3)$$

where $\tilde{q}$ has the same properties of q .

The problems (1.2)-(1.4) and (2.1)-(2.3) will be denoted by $L(p, q)$ and $\tilde{L}(p, \tilde{q})$, and spectrums of these problems will be denoted by $\sigma(p, q)$ and $\tilde{\sigma}(p, \tilde{q})$, respectively.

Theorem 2. Suppose that $\sigma(p, q) = \tilde{\sigma}(p, \tilde{q})$

and $\alpha(\pi) = 0$, then we

$$\text{get } \int_0^\pi [q(x) - \tilde{q}(x)] dx = 0.$$

Proof: Since $\sigma(p, q) = \tilde{\sigma}(p, \tilde{q})$ It follows that

$$\lambda_n \in \sigma(p, q) = \tilde{\sigma}(p, \tilde{q}) \text{ are large eigenvalues.}$$

Then, we can write from (1.4)

$$\begin{aligned} \varphi'(\pi, \lambda_n) &= -(\lambda_n - \alpha'(\pi)) \sin[\lambda_n \pi - \alpha(\pi)] + \\ &A(\pi, \pi) \cos(\lambda_n \pi) \end{aligned}$$

$$\begin{aligned} &+ B(\pi, \pi) \sin(\lambda_n \pi) + \int_0^\pi A_x(\pi, t) \cos(\lambda_n t) dt \\ &+ \int_0^\pi B_x(\pi, t) \sin(\lambda_n t) dt = 0 \end{aligned}$$

$$\begin{aligned} \tilde{\varphi}'(\pi, \lambda_n) &= -(\lambda_n - \alpha'(\pi)) \sin[\lambda_n \pi - \alpha(\pi)] \\ &+ \tilde{A}(\pi, \pi) \cos(\lambda_n \pi) \end{aligned}$$

$$\begin{aligned} &+ \tilde{B}(\pi, \pi) \sin(\lambda_n \pi) + \int_0^\pi \tilde{A}_x(\pi, t) \cos(\lambda_n t) dt \\ &+ \int_0^\pi \tilde{B}_x(\pi, t) \sin(\lambda_n t) dt = 0 \end{aligned}$$

and subtract the $\varphi'(\pi, \lambda_n)$ and $\tilde{\varphi}'(\pi, \lambda_n)$, we obtain

$$\begin{aligned} &\left[A(\pi, \pi) - \tilde{A}(\pi, \pi) \right] \cos(\lambda_n \pi) \\ &+ \left[B(\pi, \pi) - \tilde{B}(\pi, \pi) \right] \sin(\lambda_n \pi) \\ &+ \int_0^\pi \left[A_x(\pi, t) - \tilde{A}_x(\pi, t) \right] \cos(\lambda_n t) dt \\ &+ \int_0^\pi \left[B_x(\pi, t) - \tilde{B}_x(\pi, t) \right] \sin(\lambda_n t) dt \\ &= 0. \end{aligned}$$

By using Riemann Lebesgue lemma and for $\lambda_n \rightarrow \infty$ in Lemma 1, we obtain that

$A(\pi, \pi) = \tilde{A}(\pi, \pi)$. On the other hand, by Theorem 1., we get

$$\int_0^\pi [q - \tilde{q}] dx = \left(A(\pi, \pi) - \tilde{A}(\pi, \pi) \right) \cos \alpha(\pi) + \left(B(\pi, \pi) - \tilde{B}(\pi, \pi) \right) \sin \alpha(\pi).$$

Since $A(\pi, \pi) = \tilde{A}(\pi, \pi)$ and $\alpha(\pi) = 0$, we obtain $\int_0^\pi [q - \tilde{q}] dx = 0$.

Theorem 3. If the eigenvalues of the (1.2)-(1.4) Neumann boundary value problem are $\lambda_n = n$ for $n \geq 1$, and $\alpha(\pi) = 0$ then, $q(x) = p(x) = 0$ almost everywhere on $[0, \pi]$.

Proof: From (1.4)

$$\varphi'(\pi, \lambda_n) = -(\lambda_n - \alpha'(\pi)) \sin[\lambda_n \pi - \alpha(\pi)] + A(\pi, \pi) \cos(\lambda_n \pi)$$

$$+ B(\pi, \pi) \sin(\lambda_n \pi) + \int_0^\pi A_x(\pi, t) \cos(\lambda_n t) dt + \int_0^\pi B_x(\pi, t) \sin(\lambda_n t) dt$$

$$= (\alpha'(\pi) - n) \sin[n\pi - \alpha(\pi)] + A(\pi, \pi) \cos(n\pi) + B(\pi, \pi) \sin(n\pi)$$

$$+ \int_0^\pi A_x(\pi, t) \cos(nt) dt + \int_0^\pi B_x(\pi, t) \sin(nt) dt$$

$$= (\alpha'(\pi) - n) \cos(n\pi) \sin \alpha(\pi)$$

$$+ A(\pi, \pi) \cos(n\pi) + \int_0^\pi A_x(\pi, t) \cos(nt) dt$$

$$+ \int_0^\pi B_x(\pi, t) \sin(nt) dt = 0, \quad n \geq 1.$$

By using the Riemann-Lebesgue lemma, we obtain

$$(\alpha'(\pi) - n) \sin \alpha(\pi) + A(\pi, \pi) = 0.$$

Since $\alpha(\pi) = 0$, the last equality gives

$A(\pi, \pi) = 0$. By integration

$$q(x) = -p^2(x) + 2 \frac{d}{dx} \left[\begin{array}{c} A(x, x) \cos \alpha(x) \\ + \\ B(x, x) \sin \alpha(x) \end{array} \right],$$

we obtain that

$$A(x, x) = \frac{1}{\cos \alpha(x)} \left\{ \frac{1}{2} \int_0^x [q(t) + p^2(t)] dt - B(x, x) \sin \alpha(x) \right\}$$

since $\alpha(\pi) = 0$, we have

$$0 = A(\pi, \pi) = \left\{ \frac{1}{2} \int_0^\pi [q(t) + p^2(t)] dt \right\}$$

$$\text{or } \int_0^\pi q(x) dx = - \int_0^\pi p^2(x) dx.$$

Let y_1 be an eigenfunction corresponding to the $\lambda_0 = 0$ eigenvalue of the problem (1.2)-(1.4). Then,

$$\left. \begin{array}{l} -y_1'' + q(x)y_1 = 0, \quad x \in [0, \pi] \\ y_1'(0) = 0, y_1'(\pi) = 0. \end{array} \right\} \quad (2.4)$$

We claim that $y_1(0) \neq 0$ and $y_1(\pi) \neq 0$. Indeed, otherwise if $y_1(0) = 0 = y_1'(0)$ or $y_1(\pi) = 0 = y_1'(\pi)$, then, we obtain that $y_1(x) = 0$ which contradicts the fact that $y_1(x)$ is an eigenfunction corresponding to the eigenvalue $\lambda_0 = 0$. From (2.1) we obtain

$$\int_0^\pi \frac{y_1''(x)}{y_1(x)} dx = \int_0^\pi q(x) dx = - \int_0^\pi p^2(x) dx,$$

$$\frac{y_1'(x)}{y_1(x)} \Big|_0^\pi + \int_0^\pi \left[\frac{y_1'(x)}{y_1(x)} \right]^2 dx = - \int_0^\pi p^2(x) dx,$$

$$\int_0^\pi \left[\frac{y_1'(x)}{y_1(x)} \right]^2 dx = - \int_0^\pi p^2(x) dx,$$

$$\int_0^\pi \left[\left(\frac{y_1'(x)}{y_1(x)} \right)^2 + p^2(x) \right] dx = 0,$$

and we obtain a nonlinear differential equation as $(y_1')^2 + p^2(y_1)^2 = 0$ which satisfies $p = 0$

and $y_1 = 0$. Thus, $y_1(x) = k$ (k is a constant). Substituting this in the equation (2.4), we get $-k'' + q(x)k = 0$ and $q(x) = 0$ almost everywhere on $[0, \pi]$. This completes the proof.

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3. References

1. Ambarzumyan, V. A., (1929) Über eine frage der eigenwerttheorie, *Zeitschrift für Physik*, **53**, 690-695.
2. Bairamov, E., Çakar Ö. and Çelebi A.O., (1997) Quadratic pencil of Schrödinger operators with spectral singularities, Discrete spectrum and principal functions, *Journal of Mathematical Analysis and Applications*, **216**(1), 303-320.
3. Browne P. J. and Sleeman B.D., (1997) A uniqueness theorem for inverse eigenparameter dependent Sturm-Liouville problems, *Inverse Problems*, **13**, 1453-1462 .
4. Carlson R. and Pivovachik V. N., (2007) Ambarzumian's theorem for trees, *Electronic Journal of Differential Equations*, **142**, 1--9 .
5. Chakravarty N. K. and Acharyya S. K., (1988) On an extension of the theorem of V. A. Ambarzumyan, *Proceeding of the Royal Society of Edinburg*, **110A**, 79-84 .
6. Chern H. H. and Shen C. L., (1997) On the n -dimensional Ambarzumyan's theorem, *Inverse Problems*, **13**, 15-18.
7. Chern H. H., Law C. K. and Wang H. J., (2001) Extension of Ambarzumyan's theorem to general boundary conditions, *Journal of Mathematical Analysis and Applications*, **263**, 333-342.
8. Gasymov M. G. and Guseinov G. Sh., (1981) Determination of a diffusion operator from the spectral data, *Doklady Akademik Nauk Azerbaijan SSR*, **37**(2), 19-23.
9. Guseinov G.Sh., (1985) On spectral analysis of a quadratic pencil of Sturm-Liouville operators. *Soviet Mathematic Doklady*, **32**, 859-862.
10. Freiling G. and Yurko V.A., (2001) *Inverse Sturm-Liouville Problems and their Applications*, NOVA Science Publishers, New York.
11. Harrell E. M., (1987) On the extension of Ambarzumyan's inverse spectral theorem to compact symmetric spaces, *American Journal of Mathematics*, **109**, 787--795 .
12. Horvath M., (2001) On a theorem of Ambarzumyan, *Proceedings of the Royal Society of Edinburg*, **131(A)**, 899--907.
13. Koyunbakan H. and Panakhov E. S., (2007) Half inverse problem for Diffusion operators on the finite interval, *Journal of Mathematical Analysis and Applications*, **326**, 1024-1030.
14. Levitan B. M. and Gasymov M. G., (1964) Determination of a differential equation by two of its spectra, *Uspehi Mathematic Nauk*, **19**, 3--63 .
15. Jaulent M. and Jean C., (1972) The inverse s-wave scattering problem for a class of potentials depending on energy, *Communications in Mathematic Physics*, **28**, 177-220.
16. Levitan B. M., (1951) On the determination of a differential equation from its spectral function, *American Mathematic Society Translations*, **1** 253-304.
17. Nabiev I. M. , (2007) The inverse quasiperiodic problem for a diffusion operator, *Doklady Mathematics*, **76**, 527-529.
18. Pöschel J. and Trubowitz E., (1987) *Inverse spectral Theory*, Academic Press, Orlando.
19. Rundell W. and Sack E.P., (2001) Reconstruction of a radially symmetric potential from two spectral sequences, *Journal of Mathematical Analysis and Applications*, **264**, 354-381.
20. Shen C.L., (2007) On some inverse spectral problems related to the Ambarzumyan problem and the dual string of the string equation, *Inverse Problems*, **23**, 2417--2436.
21. Yang C.F. and Yang X.P., (2011), Ambarzumyan's theorem for with eigenparameter in the boundary conditions, *Acta Mathematica Scientia*, **31**(4) 1561-1568.
22. Yang C.F. and Yang X.P., (2009) Some Ambarzumyan type theorems for Dirac operators, *Inverse Problems*, **25**(9).
23. Yang C.F., Huang Z.Y. and Yang X.P., (2010), Ambarzumyan's theorems for vectorial Sturm-Liouville systems with coupled boundary conditions, *Taiwanese Journal of Mathematics*, **14**(4), 1429-1437.