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FOCAL CURVE ON TIME SCALE

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MASTER THESIS

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ABSTRACT

We introduced time scale in general then we studied time scale differential and integral, in addition we solved new examples and applications of the time scale. We defined curves, tangent vector, normal vector, binormal vector, curvatures, and focal curve then we solved examples of them. We created a new definition of curve, of tangent vector, normal vector, binormal vector, curvature and focal curve on time scale and we constructed a new equation of focal curve and focal curvature on time scale, we also constructed an equation of tangent vector, normal vector, binormal vector and curvature on time scale.

KEY WORDS: Time scale, regular Curve, osculating plane, normal plane, rectifying plane, vector-valued function, Δ-regular curve, Δ-tangent line, Δ-focal curve, Δ-focal curvature, Δ-tangent vector, Δ-normal vector, Δ-binormal vector and Δ-curvature.

ÖZET

Zaman Skalansında Focal Eğri

Genel olarak zaman skalasını tanıttık, daha sonra zaman skalasında diferansiyel ve integrali inceledik ve zaman skalasında yeni örneklerini ve uygulamalarını da çözdük. Eğrileri, teğet vektörü, normal vektörü, binormal vektörü, eğrilikleri ve fokal eğriyi tanımladık, sonra örneklerini çözdük. Teğet vektör, normal vektör, binormal vektör, eğrilik ve fokal eğriyi zaman skalasında yeniden bir tanımı oluşturduk ve zaman skalasında fokal eğri ve fokal eğrilik için yeni bir denklem inşa ettik, ayrıca teğet vektör denklemi, normal vektör, binormal vektör ve zaman skalasında eğrilikleri oluşturduk.

ANAHTAR KELİMELER: Zaman Skala**,** regüler eğri, oskülatör düzlem, normal düzlem, doğrultma düzlem, vektör fonksiyon, Δ-regüler eğri, Δ-tanjant doğru, Δ-fokal eğri, Δ-fokal eğriliği, Δ-tanjant vektör, Δ-normal vektör, Δ-binormal vektör ve Δ-eğrilik.

LIST OF SYMBOLS

$\mathbb{R}$: Set of real numbers.

$\mathbb{Z}$: Set of integer numbers.

$\mathbb{N}$: Set of natural numbers.

$\mathbb{C}$: Set of complex numbers.

$\mathbb{Q}$: Set of rational numbers.

$\mathbb{R}/\mathbb{Q}$: Set of irrational numbers.

$\mathbb{T}$: Set of Times scale.

σ: Forward jump operator.

ρ: Backward jump operator.

μ: Graininess function.

$f^{\Delta}(t)$: Delta drivative of function.

$\mathbb{T}^k$: Derived from the time scale $\mathbb{T}$.

R^n: Real number of n dimension.

τ: Tousion.

C_{α}: Focal curve.

1. INTRODUCTION

The time scale theory, recently received a lot of attention, it was make known to by Stefan Hilger in his Ph.D. thesis in 1988 (supervised by Bernd Aulbach) in order to combine discrete analysis and continuous. It is also proved and useful to be used in the mathematical modeling area of many important dynamic processing. As a result, the dynamic system theory on time scales had being developed [**1, 2**].

In this paper, we are going to study curves on time scale, which can be an arbitrary closed subset of the set of all real numbers. Therefore, our aim is to use differential part of Classical Differential Geometry on time scales calculus [**27**].

Consider that normal planes is $\triangle$-differentiable vector that is each component of the vector is $\triangle$-differentiable (i.e. normal planes) that hold first and second order $\triangle$-differentiable vectors (i.e. osculating planes). In this paper, normal and osculating planes of the curves parameterized by a compact subinterval of a time scale represented, since vector valued functions required to study $\triangle$-differentiable vectors of curves [**25**].

The focal curve C_α is defined as the centers of the osculating spheres of α. Since the center of any sphere tangent to at a point that lays on the normal plane to α at that point, the focal curve of α may be parameterized using the Frenet frame formula $(T(t), N(t), B(t))$ of as follows: $C_\alpha(t) = (\alpha(t) + cN(t) + eB(t))$, where the coefficients c and e are smooth functions that called focal curvatures of α [**28**].

The main purpose in this paper is to study focal curve combined on time scale, also tangent, normal, binormal vectors have been combine with time scales that led us to find new equations and each one has a new names.

2. BASIC DEFINITION AND THEOREM

Definition 2.1 (Time Scale)

Time Scale ($\mathbb{T}$) is an arbitrary non-empty closed subset of the real numbers **[1, 2, 7].**

Example 2.1.1

The set of the real $\mathbb{R}$, the integers $\mathbb{Z}$, the natural numbers $\mathbb{N}$, and the cantor set are examples, But the rational numbers $\mathbb{Q}$, the irrational numbers $\mathbb{R}/\mathbb{Q}$, the complex numbers $\mathbb{C}$, and the open interval between 0 and 1 (0,1), are not time scales.

Definition 2.1.2

Let $\mathbb{T}$ be a time scale , for $t \in \mathbb{T}$ we define the **forward jump operator** $\sigma : \mathbb{T} \to \mathbb{T}$ as

$$\sigma = \begin{cases} \inf\{s \in \mathbb{T} \mid t < s\}, & t < \sup \mathbb{T} \\ \sup \mathbb{T}, & t \geq \sup \mathbb{T} \end{cases}$$

and we define the **backward jump operator** $\rho : \mathbb{T} \to \mathbb{T}$ as

$$\rho = \begin{cases} \sup\{s \in \mathbb{T} \mid t > s\}, & t > \inf \mathbb{T} \\ \inf \mathbb{T}, & t \leq \inf \mathbb{T} \end{cases}$$

In this definition we get $\inf \varnothing = \sup \mathbb{T}$ and $\sup \varnothing = \inf \mathbb{T}$, where $\varnothing$ is empty set **[1, 2,7]**, its mean that :

- $\sigma(t) = t$ if $\mathbb{T}$ has a maximum t, in this case t is called **right dense**.
- $\rho(t) = t$ if $\mathbb{T}$ has minimum t, in this case t is called **left dense**.
- If $\sigma(t) = t = \rho(t)$ then t is called **dense**.
- In other case if $\sigma(t) > t$ we say that t is **right-scattered** .
- If $\rho(t) < t$ we say that t is **left-scattered** .
- If it is both left and right scattered as the same time $\rho(t) < t < \sigma(t)$ we say that t is **isolated point** .

Table 2.1.

t right-dense	$t = \sigma(t)$
t left-dense	$t = \rho(t)$
t dense	$\sigma(t) = t = \rho(t)$
t right-scattered	$t < \sigma(t)$
t left-scattered	$t > \rho(t)$
t isolated	$\sigma(t) > t > \rho(t)$

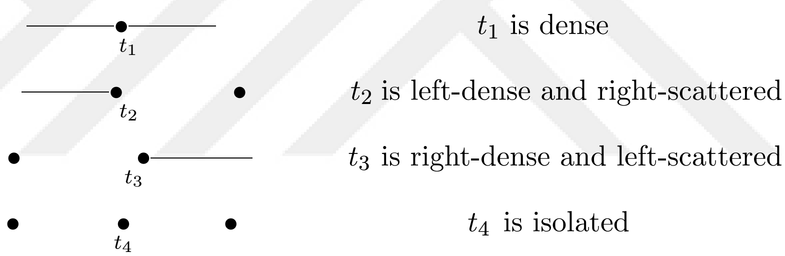

Figure 2.1. Classification of Points

Example 2.1.3

If $\mathbb{T} = [0,1] \cup \mathbb{N}$ then computation forward and backward jumps and classify each point $t \in \mathbb{T}$.

Solution:

$\mathbb{T} = [0,1] \cup \{1,2,3,4,...\}$

$\sigma(0) = 1$ and $\sigma(1) = 2, \sigma(2) = 3, \sigma(3) = 4, ..., \sigma(t) = t+1$, then we can show

$$\sigma(t) = \begin{cases} t, & \text{for } t \in [0,1) \\ t+1, & \text{for } t \in \mathbb{N} \end{cases}$$

whilst $\rho(0)=0, \rho(1)=0$ and $\rho(2)=1, \rho(3)=2, \rho(4)=3, ..., \rho(t)=t-1$ we can show

$$\rho(t)=\begin{cases} t, & \text{for } t\in[0,1] \\ t-1, & \text{for } t\in\mathbb{N}\backslash\{1\} \end{cases}$$

then 0 is left-dense and $\forall t\in(0,1)$ is dense, and t is left-scattered and right-scattered $\forall t\in\mathbb{N}$.

Definition 2.1.4

Let $\mathbb{T}$ be a time scale for $t\in\mathbb{T}$ we defined the **graininess function** $\mu:\ \mathbb{T}\to[0,\infty]$ as :

$$\mu=\sigma(t)-t$$

Example 2.1.5

For each of two time scales find σ, ρ and μ , then classify each point $t\in\mathbb{T}$.

i) $\mathbb{T}=\mathbb{R}$

ii) $\mathbb{T}=\mathbb{Z}$

Solution :

i) If $\mathbb{T}=\mathbb{R}$ then we have for any $t\in\mathbb{R}$ the forward jump operator is

$$\sigma(t)=\inf\{s\in\mathbb{R}:t<s\}=\inf\{t,t+1,t+2,...,\infty\}=\inf(t,\infty)=t$$

and the backward jump operator is

$$\rho(t)=\sup\{s\in\mathbb{R}:t>s\}=\sup\{t,t-1,t-2,...,-\infty\}=\sup(-\infty,t)=t \quad .$$

since $\sigma(t)=t=\rho(t)$, then every point $t\in\mathbb{R}$ is dense.

the graininess function is $\mu=\sigma(t)-t=t-t=0$.

ii) If $\mathbb{T}=\mathbb{Z}$ then we have for any $t\in\mathbb{Z}$ the forward jump operator is

$$\begin{aligned}\sigma(t)&=\inf\{s\in\mathbb{Z}:t<s\}=\inf\{t+1,t+2,t+3,...,\infty\}\\&=\inf(t+1,\infty)=t+1\end{aligned}$$

and the backward jump operator is

$$\rho(t) = \sup\{s \in \mathbb{Z} : t > s\} = \sup\{t-1, t-2, ..., -\infty\}$$
$$= \sup(-\infty, t-1) = t-1$$

since $\sigma(t) > t > \rho(t)$ then every point $t \in \mathbb{Z}$ is isolated.

the graininess function is $\mu = \sigma(t) - t = t+1-1 = t$.

Definition 2.1.6

Let $\mathbb{T}$ be a time scale then the set $\mathbb{T}^K$ is derived from the time scale $\mathbb{T}$ as follows:

$$\mathbb{T}^K = \begin{cases} \mathbb{T} - \{\max \mathbb{T}\}, & \text{if } \max \mathbb{T} < \infty \text{ and it has a right-scattered} \\ \mathbb{T}\ , & \text{Otherwise} \end{cases} \quad \textbf{[1, 2].}$$

Example 2.1.7

Find $\mathbb{T}^k$ for $\mathbb{T} = (0,2] \cup \{4\} \cup [5,6] \cup \{8,9\}$.

Solution:

$\max \mathbb{T} = 9 < \infty$ and $\max \mathbb{T}$ has a right-scattered then

$\mathbb{T}^K = \mathbb{T} - \{9\} =$ $(0,2] \cup \{4\} \cup [5,6] \cup \{8\}$,

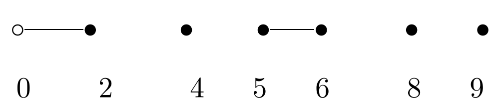

Figure 2.2. $\mathbb{T} = (0,2] \cup \{4\} \cup [5,6] \cup \{8,9\}$

Definition 2.1.8

If $f : \mathbb{T} \to \mathbb{R}$ is a function , then we define **delta-derivative** of f ($f^{\Delta}(t)$) to be a number with property, $\forall \varepsilon > 0$ there is a neighborhood $U \subset \mathbb{T}$ of t (i.e. $U = (t-\delta, t+\delta) \cap \mathbb{T}$ for some$\delta > 0$) such that $\left|[f(\sigma(t)) - f(s)] - f^{\Delta}(t)[\sigma(t) - s]\right| \le \varepsilon|\sigma(t) - s|$ for all $s \in$

$U \subset \mathbb{T}$, we call $f^{\Delta}(t)$ the **delta-derivative** (or **Hilger derivative)** at the point $t \in \mathbb{T}^K$ **[1, 2, 7].**

In the inequality by divide both side to $|\sigma(t) - s|$ we get:

$$\left| \frac{f(\sigma(t)) - f(s)}{\sigma(t) - s} - f^{\Delta}(t) \right| \leq \varepsilon \ ,$$

then we get:

$$f^{\Delta}(t) = \lim_{s \to t} \frac{f(\sigma(t)) - f(s)}{\sigma(t) - s} \ .$$

Example 2.1.9

Let $\mathbb{T}$ be a time scale and $f : \mathbb{T} \to \mathbb{R}$ by $f(t) = t^2$ then find $f^{\Delta}(t)$ for all $t \in \mathbb{T}^K$.

Solution:

$$\begin{aligned} f^{\Delta}(t) &= \lim_{s \to t} \frac{f(\sigma(t)) - f(s)}{\sigma(t) - s} \\ &= \lim_{s \to t} \frac{\sigma(t)^2 - s^2}{\sigma(t) - s} \\ &= \lim_{s \to t} \sigma(t) + s \\ &= \sigma(t) + t \end{aligned}$$

Example 2.1.10

Consider differentiable for that two cases: $\mathbb{T} = \mathbb{R}$ and $\mathbb{T} = \mathbb{Z}$.

Solution:

i) If $\mathbb{T} = \mathbb{R}$, then by (example 2.1.5) $\sigma(t) = t$

$$f^{\Delta}(t) = \lim_{s \to t} \frac{f(\sigma(t)) - f(s)}{\sigma(t) - s} = \lim_{s \to t} \frac{f(t) - f(s)}{t - s} = f'(t)$$

ii) If $\mathbb{T} = \mathbb{Z}$, then by (example 2.1.5) $\sigma(t) = t + 1$

$$\begin{aligned}
f^{\Delta}(t) &= \lim_{s\to t}\frac{f(\sigma(t))-f(s)}{\sigma(t)-s} \\
&= \lim_{s\to t}\frac{f(t+1)-f(s)}{t+1-t} \\
&= f(t+1)-f(t) \\
&= \Delta f(t)
\end{aligned}$$

Theorem 2.1.11

If $f:\mathbb{T}\to\mathbb{R}$ be a function and let $t\in\mathbb{T}^K$, then we have the following:

(i) If f is differentiable at t , then f is continuous at t .

(ii) If f is continuous at t and t is right-scattered $(\mu(t)>0)$, then f is differentiable at t with:

$$f^{\Delta}(t)=\frac{f(\sigma(t))-f(t)}{\mu(t)}$$

(iii) If f is right-dense $(\mu(t)=0)$, then f is differentiable at t with:

$$f^{\Delta}(t)=\lim_{s\to t}\frac{f(t)-f(s)}{t-s}$$

(iv) If f is differentiable at t, then

$$f(\sigma(t))=f(t)+\mu(t)f^{\Delta}(t)$$

It can be found in **[1, 2, 3].**

Example 2.1.12

For the function $f:\mathbb{T}\to\mathbb{R}$ by $f(t)=t^2, t\in\mathbb{T}=\{\frac{1}{n}:n\in\mathbb{N}\}\cup\{0\}$ use theorem 2.1.11 to find $f^{\Delta}(t)$.

Solution:

$\mathbb{T} = \{..., \frac{1}{4}, \frac{1}{3}, \frac{1}{2}, 1\} \cup \{0\}$ then $\sigma(0) = 0, \sigma(\frac{1}{4}) = \frac{1}{3}, \sigma(\frac{1}{3}) = \frac{1}{2}, \sigma(\frac{1}{2}) = 1, ..., \sigma(t) = \frac{t}{1-t}$

$\mu(t) = \sigma(t) - t = \frac{t}{1-t} - t = \frac{t^2}{1-t} > 0$

and f is continuous then by theorem 2.1.11 we use

$$
\begin{aligned}
f^{\Delta}(t) &= \frac{f(\sigma(t)) - f(t)}{\mu(t)} = \frac{\sigma^2(t) - t^2}{\frac{t^2}{1-t}} \\
&= \frac{\frac{t^2}{(1-t)^2} - t^2}{\frac{t^2}{1-t}} = \frac{\frac{t^2(1-(1-t)^2)}{(1-t)^2}}{\frac{t^2}{1-t}} \\
&= \frac{1-(1-t)^2}{1-t} = \frac{1-1+2t-t^2}{1-t} \\
&= \frac{t(2-t)}{1-t}
\end{aligned}
$$

Theorem 2.1.13

Let $\mathbb{T}$ be a time scale and $f, g : \mathbb{T} \to \mathbb{R}$ are differentiable at $t \in \mathbb{T}^k$ then:

(i) $f + g : \mathbb{T} \to \mathbb{R}$ is differentiable at $t \in \mathbb{T}^k$ with

$$(f+g)^{\triangle}(t) = f^{\triangle}(t) + g^{\triangle}(t) \cdot$$

(ii) $\alpha f : \mathbb{T} \to \mathbb{R}$ is differentiable at $t \in \mathbb{T}^k$ with

$(\alpha f)^{\triangle}(t) = \alpha f^{\triangle}(t)$, if α be any constant.

(iii) $f.g : \mathbb{T} \to \mathbb{R}$ is differentiable at $t \in \mathbb{T}^k$ with

$$
\begin{aligned}
(f.g)^{\triangle}(t) &= f^{\triangle}(t)g(t) + f((\sigma(t))g^{\triangle}(t) \\
&= f(t)g^{\triangle}(t) + f^{\triangle}(t)g(\sigma(t))
\end{aligned}
$$

(iv) If $f(t) \cdot f((\sigma(t)) \neq 0$, then $\frac{1}{f}$ is differentiable at $t \in \mathbb{T}^k$ with

$$\left(\frac{1}{f}\right)^{\triangle}(t) = -\frac{f^{\triangle}(t)}{f(t)\cdot f((\sigma(t))}$$

(v) $g(t)\cdot g((\sigma(t))\neq 0$, then $\dfrac{f}{g}$ is differentiable at $t\in\mathbb{T}^k$ with

$$\left(\frac{f}{g}\right)^{\triangle}(t) = \frac{f^{\triangle}(t)g(t)-f(t)g^{\triangle}(t)}{g(t)\cdot g((\sigma(t))}$$

It can be found in **[1, 2, 3].**

Example 2.1.14

If $f:\mathbb{T}\to\mathbb{R}$ and $g:\ \mathbb{T}\to\mathbb{R}$ are two functions by $f(t)=\sigma(t)$ and $g\,(t)=t^2$ for $t\in\mathbb{T}=\{\frac{n}{2}:n\in\mathbb{N}_0\}$ then find the following:

(i) $(f+g)^{\Delta}(t)$.

(ii) $(f.g)^{\triangle}(t)$.

(iii) $\left(\dfrac{1}{f}\right)^{\triangle}(t)$.

Solution:

$\mathbb{T}=\{0,\frac{1}{2},1,\frac{3}{2},2,\frac{5}{2},3,...\}$

then $\sigma(0)=\frac{1}{2},\sigma(\frac{1}{2})=1,\sigma(1)=\frac{3}{2},\sigma(\frac{3}{2})=2,...,\sigma(t)=\dfrac{2t+1}{2}$

$\mu(t)=\sigma(t)-t=\dfrac{2t+1}{2}-t=\dfrac{2t+1+2t}{2}=\dfrac{4t+1}{2}>0$

and f is continuous then by theorem 2.1.11

$$\begin{aligned}
f^{\Delta}(t) &= \frac{f(\sigma(t))-f(t)}{\mu(t)}\\
&= \frac{f(\frac{2t+1}{2})-f(t)}{\frac{4t+1}{2}} = \frac{\frac{2\sigma(t)+1}{2}-\frac{2t+1}{2}}{\frac{4t+1}{2}}\\
&= \frac{\frac{2t+2}{2}-\frac{2t+1}{2}}{\frac{4t+1}{2}} = \frac{2t+2-2t+1}{4t+1}\\
&= \frac{1}{4t+1}
\end{aligned}$$

$$
\begin{aligned}
g^{\triangle}(t) &= \frac{f(\sigma(t))-f(t)}{\mu(t)}\\
&= \frac{f(\frac{2t+1}{2})-f(t)}{\frac{4t+1}{2}} = \frac{(\frac{2t+1}{2})^2-t^2}{\frac{4t+1}{2}}\\
&= \frac{\frac{4t^2+4t+1}{4}-t^2}{\frac{4t+1}{2}} = \frac{\frac{4t^2+4t+1-4t^2}{4}}{\frac{4t+1}{2}}\\
&= \frac{\frac{4t+1}{4}}{\frac{4t+1}{2}}\\
&= \frac{1}{2}
\end{aligned}
$$

(i) By theorem 2.1.13 $(f+g)^{\Delta}(t) = f^{\triangle}(t) + g^{\triangle}(t)$

$$
\begin{aligned}
&= \frac{1}{4t+1} + \frac{1}{2}\\
&= \frac{2+4t+1}{8t+2}\\
&= \frac{4t+3}{8t+2}
\end{aligned}
$$

(ii) By theorem 2.1.13 $(f.g)^{\triangle}(t) = f(t).g^{\triangle}(t) + f^{\triangle}(t).g(\sigma(t))$

$$
\begin{aligned}
&= (\frac{2t+1}{2}).\frac{1}{2} + (\frac{1}{4t+1}).(\frac{2t+1}{2})^2\\
&= \frac{2t+1}{4} + \frac{4t^2+4t+1}{16t+4}\\
&= \frac{8t^2+6t+1+4t^2+4t+1}{16t+4}\\
&= \frac{12t^2+10t+2}{16t+4}\\
&= \frac{4t^2+4t+1}{8t+2}
\end{aligned}
$$

(iii) By theorem 2.1.13 $\left(\frac{1}{f}\right)^{\triangle}(t) = -\frac{f^{\triangle}(t)}{f(t).f((\sigma(t))}$

$$= -\frac{\frac{1}{4t+1}}{(\frac{2t+1}{2}).(\frac{2t+2}{2})}$$

$$= \frac{\frac{1}{4t+1}}{\frac{2t^2+3t+1}{2}}$$

Definition 2.1.15

A function $f : \mathbb{T} \to \mathbb{R}$ is called **right-dense continuous** or (**rd-continuous**) provided it is continuous at right-dense point in $\mathbb{T}$ and left-sided limits exist at left-dense points in $\mathbb{T}$ [**1**].

Theorem 2.1.16

(i) If f is continuous then f is rd-continuous.

(ii) The forward jump operator $(\sigma(t))$ is rd-continuous.

It can be found in [**1, 2, 3**].

Definition 2.1.17

A function $F : \mathbb{T} \to \mathbb{R}$ is called a **$\triangle$-antiderivative** of $f : \mathbb{T} \to \mathbb{R}$ if satisfy $F^{\triangle}(t) = f(t)$ for all $t \in \mathbb{T}^k$ then we can define the **infinite $\triangle$-integral** of f

$$\int f(s) \triangle s = F(t) - c \quad \text{when } c \text{ is arbitrary constant of } F.$$

We define the **Cauchy $\triangle$-integral** of f from a to b of f

$$\int_a^b f(s) \triangle s = F(b) - F(a) \ , \qquad \forall a, b \in \mathbb{T} \quad [\mathbf{1,\ 2,\ 6}].$$

Example 2.1.18

Show that if $\mathbb{T}=\mathbb{Z}$, $a\neq 1$ then $\int a^t \triangle t = \dfrac{a^t}{a-1} + c$, where c is arbitrary constant.

Solution:

Therefor $\mathbb{T}=\mathbb{Z}$ thus $f^{\triangle}(t) = \triangle f(t)$ then $\left(\dfrac{a^t}{a-1}\right)^{\triangle} = \triangle\left(\dfrac{a^t}{a-1}\right) = \dfrac{a^{t+1}-a^t}{a-1} = a^t$,

so we get

$$\int a^t \triangle t = \frac{a^t}{a-1} + c \ .$$

Example 2.1.19

Show that if $\mathbb{T}=\mathbb{R}$, $k\neq -1$ then $\int (t+s)^k \triangle t = \dfrac{(t+s)^{k+1}}{k+1} + c$, where c is arbitrary constant.

Solution:

Therefor $\mathbb{T}=\mathbb{R}$ thus $f^{\triangle}(t) = f^{'}(t)$ then $\left(\dfrac{(t+s)^{k+1}}{k+1}\right)^{\triangle} = \left(\dfrac{(t+s)^{k+1}}{k+1}\right)^{'} = (t+s)^k$,

so we get $\displaystyle\int (t+s)^k \triangle t = \frac{(t+s)^{k+1}}{k+1} + c$.

Table 2.2.

Time scale $\mathbb{T}$	$\mathbb{R}$	$\mathbb{Z}$
Forward jump operator $\sigma(t)$	t	$t+1$
Backward jump operator $\rho(t)$	t	$t-1$
Graininess $\mu(t)$	0	1
Derivative $f^{\triangle}(t)$	$f^{'}(t)$	$\triangle f(t)$
Integral $\int_a^b f(t) \triangle t$	$\int_a^b f(t)dt$	$\sum_{t=1}^{b-1} f(t)$

Theorem 2.1.20

If $a,b,c \in \mathbb{T}$,$\alpha \in \mathbb{R}$, $f:\mathbb{T} \to \mathbb{R}$ and $g:\mathbb{T} \to \mathbb{R}$ then

(i) $\int\limits_a^b [f(t)+g(t)] \triangle t = \int\limits_a^b f(t) \triangle t + \int\limits_a^b g(t) \triangle t$

(ii) $\int\limits_a^b (\alpha f)(t) \triangle t = \alpha \int\limits_a^b f(t) \triangle t$

(iii) $\int\limits_a^b f(t) \triangle t = \int\limits_b^c f(t) \triangle t + \int\limits_a^b f(t) \triangle t$

(iv) $\int\limits_a^b f(t) \triangle t = -\int\limits_b^a f(t) \triangle t$

(v) $\int\limits_b^b f(t) \triangle t = 0$

(vi) $\int\limits_a^b f(\sigma(t)).g^{\triangle}(t) \triangle t = (f.g)(b) - (f.g)(a) - \int\limits_a^b f^{\triangle}(t).g(t) \triangle t$

(vii) $\int\limits_a^b f(t).g^{\triangle}(t) \triangle t = (f.g)(b) - (f.g)(a) - \int\limits_a^b f^{\triangle}(t).g(\sigma(t)) \triangle t$

It can be found in [**1, 2, 3**].

Definition 2.1.21

If $1+\mu(t)p(t) \neq 0$ then the function $p:\mathbb{T} \to \mathbb{R}$ is called regressive function for all $t \in \mathbb{T}$, and the exponential function on time scale is

$$e_p(t,s) = \exp\left(\int\limits_s^t \xi_{\mu(u)}(p(u)) \triangle u\right) \text{ for all } s,t,u \in \mathbb{T},$$

where the cylinder transformation $\xi_{\mu(u)} = \begin{cases} \frac{\log(1+u\mu)}{\mu}, & \text{if } \mu \neq 0 \\ z, & \text{if } \mu = 0 \end{cases}$ [**1, 2, 4, 7**].

We note that $e_0(t,s)=1$ and $e_p(t,t)=1$.

Table 2.3.

$\mathbb{T}$	s	$p(t)$	$e_p(t,s)$
$\mathbb{R}$	0	1	e^t
$\mathbb{Z}$	0	1	2^t

Definition 2.1.22

If μp^2 is regressive then we can define cosine and sine function on time scale by

$$\cos_p(t,s) = \frac{e_{ip}(t,s)+e_{-ip}(t,s)}{2} \quad \text{and} \quad \sin_p(t,s) = \frac{e_{ip}(t,s)-e_{-ip}(t,s)}{2i} \quad \textbf{[1, 2, 4].}$$

Theorem 2.1.23

If μp^2 is regressive then we have

(1) $\cos_p^{\triangle}(t,s) = -p\sin_p(t,s)$

(2) $\sin_p^{\triangle}(t,s) = p\cos_p(t,s)$

(3) $\sin_p^2(t,s) + \cos_p^2(t,s) = e_{\mu p^2}(t,s)$

(4) $e_{ip}(t,s) = \cos_p(t,s) + i\sin_p(t,s)$

It can be found in **[1, 2, 3].**

Definition 2.1.24

If $-\mu p^2$ is regressive then we can define cosine hyperbolic and sin hyperbolic function on time scale by

$$\cosh_p(t,s) = \frac{e_p(t,s)+e_{-p}(t,s)}{2} \quad \text{and} \quad \sinh_p(t,s) = \frac{e_p(t,s)-e_{-p}(t,s)}{2} \quad \textbf{[1, 2, 4].}$$

Theorem 2.1.25

If $-\mu p^2$ is regressive then we have

(1) $\cosh_p^{\triangle}(t,s) = p\sinh_p^{\triangle}(t,s)$

(2) $\sinh_p^{\triangle}(t,s) = p\cosh_p^{\triangle}(t,s)$

(3) $\sinh_p^2(t,s) - \cosh_p^2(t,s) = e_{-\mu p^2}(t,s)$

(4) $e_p(t,s) = \cosh_p(t,s) + \sinh_p(t,s)$

It can be found in **[1, 2, 3].**

Definition 2.2. (Curves)

A curve is a continuous mapping $\alpha : I \to R^n$, where I is any interval of real line R into R^n. We write a Curve of α for parameter $t \in I$ as

$$\alpha(t) = (\alpha_1(t)\ ,\ \alpha_2(t)\ , ...,\ \alpha_n(t))$$

and α is differentiable $\iff$ a_i is differentiable for $i = 1, ..., n$ **[9, 10, 11, 13].**

Definition 2.2.1

The differential of Curve (α) is called the **velocity** of α with the function as $\alpha' : I \to R^n$ and given by

$v(t) = \alpha'(t) = (a_1'(t), a_2'(t), ..., a_n'(t))$ and the **acceleration** of α is $v(t)' = \alpha''(t)$.

Definition 2.2.2

Let $\alpha : I \to R^n$ be a smooth curve, it is said to be **regular** if $\alpha'(t) \neq 0$ for all $t \in I$. If $\left|\alpha'(t)\right| = 1$ then α have **unit-speed [9, 10].**

Definition 2.2.3

Let $\alpha : I \to R^n$ be a curve then the length of α is defined by

$$L(\alpha) = \int_a^b \left|\alpha'(t)\right| dt\ , \qquad \text{for } a < t < b \quad \textbf{[9, 10].}$$

Definition 2.2.4

A **tangent line** is a line that touches a curve at a point without crossing over. Formally, it is a line which intersects a differentiable curve at a point p where the slope of the curve equals the slope of the line. The equation for the tangent line at that point p is $y - y_1 = \frac{dy}{dx}(x - x_1)$, where $\frac{dy}{dx}$ is a slope of the curve and (x_1, y_1) is a point p.

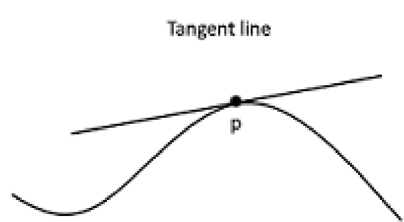

Figure 2.3. Tangent line on Curve

Definition 2.2.5

Let $\alpha : I \to R^3$ be a regular curve then the **tangent vector** along α is defined by

$$T(t) = \frac{\alpha'(t)}{|\alpha'(t)|}$$

we note that if α have unit-speed (i.e. $\left|\alpha'(t)\right| = 1$) then $T(t) = \alpha'(t)$ and we call T is **unit tangent vector.**

Definition 2.2.6

Let α any regular curve then the **curvature** is defined by

$$k = \frac{\left|T'(t)\right|}{|\alpha'(t)|} = \frac{\left|\alpha''(t)\right|}{|\alpha'(t)|} \quad \text{or} \quad k = \frac{\left|\alpha'(t) \times \alpha''(t)\right|}{|\alpha'(t)|^3} \; .$$

Definition 2.2.7

Let $\alpha : I \to R^3$ be any regular curve then the **normal vector** along α is defined by

$$N(t) = \frac{T'(t)}{k}$$

also we can defined normal vector as $N(t) = B(t) \times T(t)$, where B is binormal vector.

Definition 2.2.8

Let $\alpha : I \to R^3$ be any regular curve then the **binormal vector** defined by

$$B(t) = T(t) \times N(t) \; \cdot$$

Definition 2.2.9

The set $\{ \; T \; , \; N \; , \; B \; \}$ is called the **frenet frame** along α .

We note that of unit speed regular curve $|T| = |N| = |B| = 1$ and $TN = NB = BT = 0$ $\cdot$

Definition 2.2.10

Let α be any regular curve parameterized by arc length t, then the **torsion** of α is defined by

$$\tau(t) = \frac{-B'(t) \cdot N(t)}{\left|\alpha'(t)\right|} \; .$$

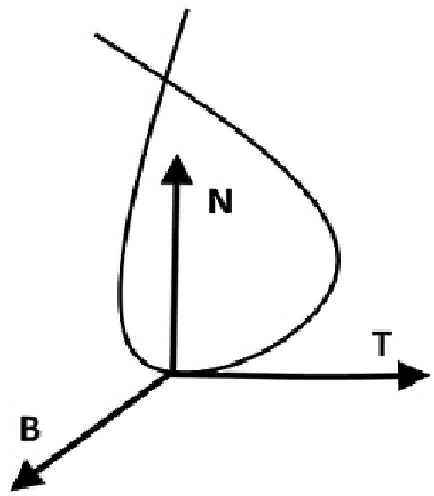

Figure 2.4. Frenet frame along curve

Definition 2.2.11

For a regular curve $\alpha : I \to R^3$, then the focal curve of the space consists of the centres of its osculating hyperspheres. This curve parametrized in terms of the Frenet frame as

$$C_\alpha(t) = \alpha(t) + cN(t) + eB(t) \ ,$$

where $c = \frac{1}{k}$ is the first focal curvature and $e = \frac{c'}{\tau}$ is the second focal curvature.

Note that if $\tau = 0$ then we don't have second focal curvature [**28, 29, 33**].

Example 2.2.12

For the mystery curve $\alpha(t) = \left(\frac{1}{\sqrt{2}}\cos t, \sin t, \frac{1}{\sqrt{2}}\cos t\right)$ for $0 < t < 2\pi$, find the focal curve.

Solution:

$$
\begin{aligned}
\alpha'(t) &= \left(\frac{-1}{\sqrt{2}}\sin t, \cos t, \frac{-1}{\sqrt{2}}\sin t\right) \\
\alpha''(t) &= \left(\frac{-1}{\sqrt{2}}\cos t, -\sin t, \frac{-1}{\sqrt{2}}\cos t\right) \\
\left|\alpha'(t)\right| &= \sqrt{\left(\frac{-1}{\sqrt{2}}\sin t\right)^2 + (\cos t)^2 + \left(\frac{-1}{\sqrt{2}}\sin t\right)^2} \\
&= \sqrt{\frac{1}{2}\sin^2 t + \cos^2 t + \frac{1}{2}\sin^2 t} \\
&= \sqrt{\sin^2 t + \cos^2 t} = \sqrt{1} = 1
\end{aligned}
$$

so the mystery curve have unit speed.

The length of the mystery is

$$
L(\alpha) = \int_0^{2\pi} \left|\alpha'(t)\right| dt = \int_0^{2\pi} dt
$$
$$
= 2\pi - 0 = 2\pi
$$

so the tangent vector is

$$
\begin{aligned}
T(t) &= \frac{\alpha'(t)}{|\alpha'(t)|} = \frac{\alpha'(t)}{1} \\
&= \left(\frac{-1}{\sqrt{2}}\sin t, \cos t, \frac{-1}{\sqrt{2}}\sin t\right)
\end{aligned}
$$

now the curvature is

$$
\begin{aligned}
k &= k = \left|T'(t)\right| = \left|\alpha''(t)\right| \\
&= \sqrt{(\frac{-1}{\sqrt{2}}\cos t)^2 + (-\sin t)^2 + (\frac{-1}{\sqrt{2}}\cos t)^2} \\
&= \sqrt{\frac{1}{2}\cos^2 t + \sin^2 t + \frac{1}{2}\cos^2 t} \\
&= \sqrt{\cos^2 t + \sin^2 t} = \sqrt{1} \\
&= 1
\end{aligned}
$$

then the normal vector along α is

$$
\begin{aligned}
N(t) &= \frac{T(t)^{'}}{k} = T(t)^{'} \\
&= \left(\frac{-1}{\sqrt{2}}\cos t, -\sin t, \frac{-1}{\sqrt{2}}\cos t\right)
\end{aligned}
$$

and the binormal vector is

$$
\begin{aligned}
B(t) &= T(t) \times N(t) \\
&= \begin{vmatrix} e_1 & e_2 & e_3 \\ \frac{-1}{\sqrt{2}}\sin t & \cos t & \frac{-1}{\sqrt{2}}\sin t \\ \frac{-1}{\sqrt{2}}\cos t & -\sin t & \frac{-1}{\sqrt{2}}\cos t \end{vmatrix} \\
&= \left(\frac{-1}{\sqrt{2}}, 0, \frac{1}{\sqrt{2}}\right) \\
B^{'}(t) &= (0,0,0) \\
\tau &= \frac{B^{'}(t) \cdot N(t)}{|\alpha^{'}(t)|} \\
&= \frac{(0,0,0) \cdot \left(\frac{-1}{\sqrt{2}}\cos t, -\sin t, \frac{-1}{\sqrt{2}}\cos t\right)}{1} \\
&= 0
\end{aligned}
$$

the first focal curvature is $c = \frac{1}{k} = 1$,

then we can find focal curve

$$
\begin{gathered}
C_\alpha(t) = \alpha(t) + cN(t) \\
= \left(\frac{1}{\sqrt{2}}\cos t, \sin t, \frac{1}{\sqrt{2}}\cos t\right) + \left(\frac{-1}{\sqrt{2}}\cos t, -\sin t, \frac{-1}{\sqrt{2}}\cos t\right) \\
= (0,0,0)
\end{gathered}
$$

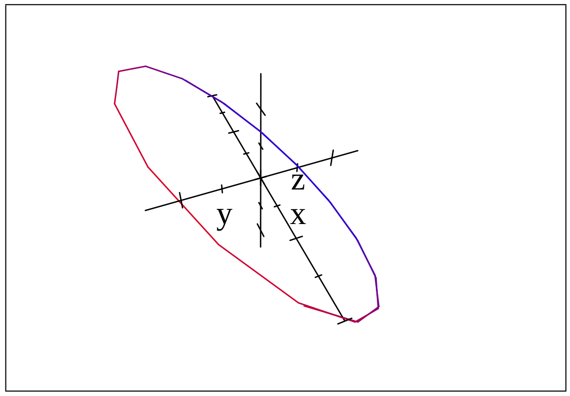

Figure 2.5. A mysterycurve

Definition 2.2.13

Let $\alpha : I \to R^3$ be a curve then the plane spanned by the tangent(T) and normal(N) vectors is called **osculating plane,** and we can determined by

$$B(t_0) \cdot (x, y, z) = B(t_0) \cdot \alpha(t_0) \ , \quad \text{when } t_0 \in I \ .$$

Definition 2.2.14

Let $\alpha : I \to R^3$ be a curve then the plane spanned by the normal(N) and binormal(B) vectors is called **normal plane,** and we can determined by

$$T(t_0) \cdot (x, y, z) = T(t_0) \cdot \alpha(t_0) \ , \quad \text{when } t_0 \in I \ .$$

Definition 2.2.15

Let $\alpha : I \to R^3$ be a curve then the plane spanned by the binormal(B) and tangent(T) vectors is called **rectifying plane,** and we can determined by

$$N(t_0) \cdot (x, y, z) = N(t_0) \cdot \alpha(t_0) \ , \quad \text{when } t_0 \in I \ .$$

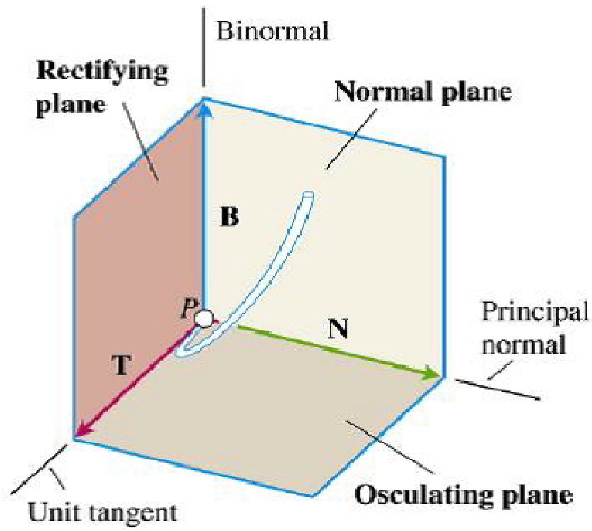

Figure 2.6. Osculating, normal and rectifying planes

Example 2.2.16

For curve $\alpha(t) = (\sin t, \cos t, t)$, find osculating, normal and rectifying planes at $t = \pi$

.

Solution:

$$\alpha^{'}(t) = (\cos t, -\sin t, 1)$$
$$\left|\alpha^{'}(t)\right| = \sqrt{2}$$
$$T(t) = \frac{\alpha^{'}(t)}{|\alpha^{'}(t)|} = \left(\frac{1}{\sqrt{2}}\cos t, -\frac{1}{\sqrt{2}}\sin t, \frac{1}{\sqrt{2}}\right)$$
$$T^{'}(t) = \left(-\frac{1}{\sqrt{2}}\sin t, -\frac{1}{\sqrt{2}}\cos t, 0\right)$$
$$\left|T^{'}(t)\right| = \sqrt{\frac{1}{2}\sin^2 t + \frac{1}{2}\cos^2 t} = \frac{1}{\sqrt{2}}$$
$$k = \frac{\left|T^{'}(t)\right|}{|\alpha^{'}(t)|} = \frac{\frac{1}{\sqrt{2}}}{\sqrt{2}} = \frac{1}{2}$$
$$N(t) = \frac{T^{'}(t)}{k} = \left(-\sqrt{2}\sin t, -\sqrt{2}\cos t, 0\right)$$
$$B(t) = T(t) \times N(t) = (\cos t, -\sin t, -1)$$

so we determine α, T, N and B at $\quad t=\pi$

$$\alpha(\pi)=(0,-1,\pi)$$
$$T(\pi)=\left(-\frac{1}{\sqrt{2}},0,\frac{1}{\sqrt{2}}\right)$$
$$N(\pi)=(0,\sqrt{2},0)$$
$$B(\pi)=(-1,0,-1)$$

for osculating plane $\quad B(\pi)\cdot(x,y,z)=B(\pi)\cdot\alpha(\pi)$

$$(-1,0,-1)\cdot(x,y,z)=(-1,0,-1)\cdot(0,-1,\pi)$$
$$-x+0y-z=-\pi \implies x+y=\pi \quad ,$$

for normal plane $\quad T(\pi)\cdot(x,y,z)=T(\pi)\cdot\alpha(\pi)$

$$\left(-\frac{1}{\sqrt{2}},0,\frac{1}{\sqrt{2}}\right)\cdot(x,y,z)=\left(-\frac{1}{\sqrt{2}},0,\frac{1}{\sqrt{2}}\right)\cdot(0,-1,\pi)$$
$$-\frac{1}{\sqrt{2}}X+0y+\frac{1}{\sqrt{2}}z=\frac{\pi}{\sqrt{2}} \implies -x+z=\pi \quad ,$$

and for rectifying plane $\quad N(\pi)\cdot(x,y,z)=N(\pi)\cdot\alpha(\pi)$

$$(0,\sqrt{2},0)\cdot(x,y,z)=(0,\sqrt{2},0)\cdot(0,-1,\pi)$$
$$\sqrt{2}y=\sqrt{2}\pi \implies y=\pi \quad .$$

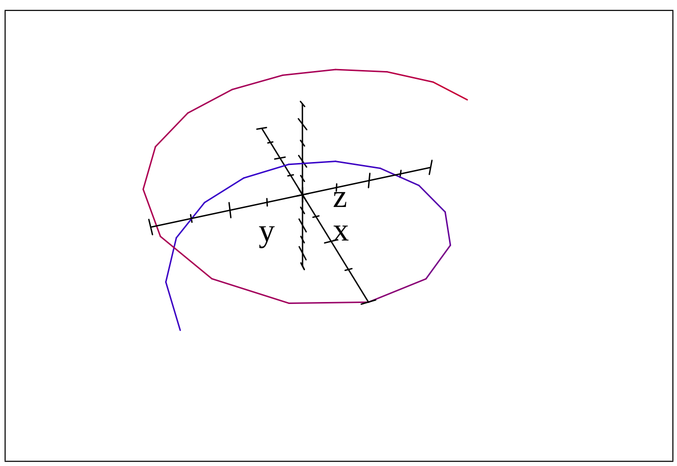

Figure 2.7. $\alpha(t)$=$(\sin t,\cos t,t)$

Definition 2.2.17

A **vector-valued function**, also referred to as a **vector function**, is a function where the domain is a subset of the real numbers and the range is a multidimensional vector.

The dimension of the domain is not defined by the dimension of the range.

In two dimensions we get **plane curve** it write as

$r(t) = x(t)i + y(t)j = (x(t), y(t))$

and in three dimensions we get **space curve,** it write as

$r(t) = x(t)i + y(t)j + z(t)k = (x(t), y(t), z(t))$, where x, y and z are the component functions [**15, 16, 17, 18**].

Example 2.2.18

Find the velocity($v(t)$) and acceleration($a(t)$) of the vector valued function,

$r(t) = 2\cos t$ i$+2\sin t$ j$+5\cos^2 t$ k .

Solution:

The velocity is

$$v(t) = r^{'}(t) = -2\sin t \text{ i} + 2\cos t \text{ j} - 10\cos t \sin t \text{ k}$$
$$= -2\sin t \text{ i} + 2\cos t \text{ j} - 5\sin 2t \text{ k}$$

and the acceleration is

$$a(t) = v^{'}(t) = r^{''}(t) = -2\cos t \text{ i} - 2\sin t \text{ j} - 10\cos 2t \text{ k} .$$

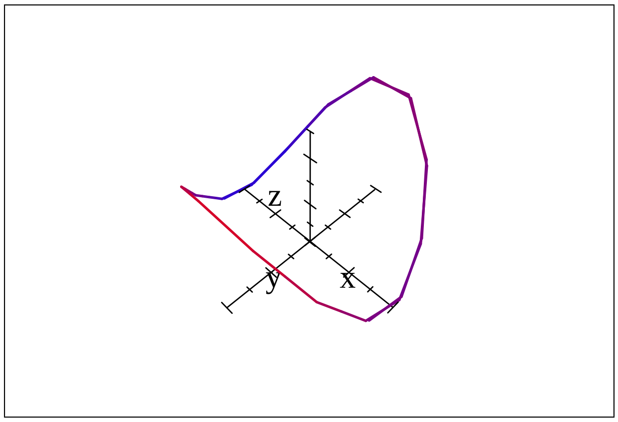

Figure 2.8. $r(t)$=$(2\cos t, 2\sin t, 5\cos^2 t)$

Definition 2.2.19

Let $r(t)$ be a vector-valued function then we define the **tangent line** to $r(t)$ at a point t_0 to be the line parallel to the derivative $r^{'}(t_0)$.

The equation for tangent line is

$$R(t) = r(t_0) + v(t_0)t \ \cdot$$

Example 2.2.20

Find the tangent line at $t = \pi$ of a circular helix with the equation

$r(t) = a\cos t \text{ i}+a\sin t \text{ j}+ct \text{ k}$.

Solution:

$$r^{'}(t) = -a\sin t \text{ i}+a\cos t \text{ j}+c \text{ k}$$

we calculate the tangent line at $t = \pi$

$$r(\pi) = -a \text{ i}+\pi c \text{ k}$$

$$r^{'}(\pi) = -a \text{ j}+c \text{ k}$$

then

$$R(t) = r(\pi) + r^{'}(\pi)$$

$$R(t) = (-a \text{ i}+\pi c \text{ k}) + (-a \text{ j}+c \text{ k})t$$

$$= -a \text{ i}-at \text{ j}+c(t+\pi) \text{ k}$$

Definition 2.2.21

Let X be a set. A function $d : X \times X \to \mathbb{R}$ is called a **metric** if satisfies the following three properties:

1. $d(x, y) \geq 0$ for all $x, y \in X$ and $d(x, y) = 0 \iff x = y$, (distance are non-negative,and distance is zero only distance from x to x itself)

2. $d(x, y) = d(y, x)$ for all $x, y \in X$, (the distance is symmetric function)

3. $d(x, y) \leq d(x, z) + d(z, y)$, for all $x, y, z \in X$, (the distance satisfy the triangle inequality).

Then a pair (X, d) is a set and called a **metric space.**

It can be found in **[20, 22, 23, 24].**

Example 2.2.22

The set of real numbers $\mathbb{R}$ with the distance function $d(x, y) = |x - y|$ is a metric space and the set of complex numbers $\mathbb{C}$ with the distance function $d(x, y) = |x - y|$ is metric space.

3. REGULAR CURVE ON TIME SCALE

Definition 3.1

Let n be a fixed and $\mathbb{T}_i$ denote a time scale for each $i \in \{1, 2, ..., n\}$, and let

$$\Lambda^n = \mathbb{T}_1 \times \mathbb{T}_2 \times ... \times \mathbb{T}_n = \{t = (t_1, ..., t_n) : t_i \in \mathbb{T} \;\; \forall i \in \{1, 2, ..., n\}\} \; .$$

We call Λ^n an n-**dimensional time scale**, Λ^n is also a complete metric space with

$$d(x, y) = \sqrt{\sum |x_i - y_i|^2}\,, \quad \forall x, y \in \Lambda^n \quad \textbf{[25, 26, 30].}$$

Let a time scale be change with parameter t in an interval $[a, b]$.

If to each value $t \in [a, b]$ we assign a vector $r(t)$, then we say that the vector-valued function $r(t)$ with argument $t \in [a, b]$ is given.

Assume that coordinates $x_1, x_2, ..., x_n$ are fixed, then the representation of vector-valued function $r(t)$ is equivalent to scalar functions $x_1(t), x_2(t), ..., x_n(t)$, that is

$r(t) = \{x_1(t), x_2(t), ..., x_n(t)\}$ **[25, 26, 30].**

Definition 3.2

A vector $r(t_0)$ is called **the limit of the vector-valued function** $r(t)$ as $t \to t_0$ if the length of the vector $r(t) - r(t_0)$ tends to zero as $t \to t_0$, here we write

$$\lim_{t \to t_0} r(t) = r(t_0) \; .$$

It is clear that the vector-valued function $r(t)$ has a limit if and only if each one of the functions $x_1(t), x_2(t), ..., x_n(t)$ has a limit at $t \to t_0$ **[25].**

Definition 3.3

$\triangle$-Derivative of a vector-valued function can be obtained by $\triangle$-differentiating components $x_1(t), x_2(t), ..., x_n(t)$ of $r(t)$, and defined by

$$r^{\triangle}(t) = \{x_1^{\triangle}(t), x_2^{\triangle}(t), ..., x_n^{\triangle}(t)\} \; .$$

Precisely, for $\triangle$-derivative $r^{\triangle}(t)$ of the vector-valued function $r(t)$ we can defined by this limit

$$\lim_{s \to t} \frac{r(\sigma(t)) - r(s)}{\sigma(t) - s} \;\; , \;\; \text{where backward operator jump}(\sigma(t)) \neq s$$

and if this limit exists then $r(t)$ is called **$\triangle$-differentiable of vector-valued function [25, 26].**

Proposition 3.4

Let $r_1(t)$ and $r_2(t)$ be vector-valued function then

(i) $(r_1(t)+r_2(t))^{\triangle} = r_1^{\triangle}(t)+r_2^{\triangle}(t)$

(ii) $(r_1(t).r_2(t))^{\triangle} = r_1^{\triangle}(t).r_2(t)+r_1^{\sigma}(t).r_2^{\triangle}$

The $\triangle$-differentiation of the vector products of vector-valued functions is computed by consecutive differentiation of the cofactors.

(iii) $(r_1(t)\times r_2(t))^{\triangle} = \ r_1^{\triangle}(t)\times r_2(t)+r_1^{\sigma}(t)\times r_2^{\triangle} = r_1^{\triangle}(t)\times r_2^{\sigma}(t)+r(t)\times r_2^{\triangle}(t)$,

where $\times$ be Euclidean vector product.

Definition 3.5

Let $\mathbb{T}$ be a time scale, a **$\triangle$-regular curve** $\Gamma : \mathbb{T} \rightarrow R^3$ is defined as a mapping

$\Gamma(t) = (\Gamma_1(t), \Gamma_2(t), \Gamma_3(t))$, where $x = \Gamma_1(t)$, $y = \Gamma_2(t)$, $z = \Gamma_3(t)$, $t \in [a,b] \subset \mathbb{T}$

where Γ_1, Γ_2 and Γ_3 are real-valued functions defined on $[a,b]$, that are $\triangle$-differentiable on $[a,b]^k$with rd-continuous satisfied

$$\left|\Gamma_1^{\triangle}(t)\right|^2 + \ \left|\Gamma_2^{\triangle}(t)\right|^2 + \ \left|\Gamma_3^{\triangle}(t)\right|^2 \neq 0 \ , \quad t \in [a,b]^k \ \textbf{[25, 27, 30].}$$

Definition 3.6

Let Γ be $\triangle$-regular curve. A line $\mathcal{L}$ is called **$\triangle$-tangent line** to the curve Γ at the point P_0 if the following held

(i) $\mathcal{L}$ passing through the point $P_0^{\sigma} = (\Gamma_1(\sigma(t_0)), \Gamma_2(\sigma(t_0)), \Gamma_3(\sigma(t_0))$.

(ii) If $P_0 = (\Gamma_1(t_0), \Gamma_2(t_0), \Gamma_3(t_0))$ is not an isolated point of the curve Γ then

$$\lim_{P\rightarrow P_0} \frac{d(P,\mathcal{L})}{d(P,P_0)} \quad \text{exists , where } P \neq P_0 \ ,$$

P is the moving point of the curve Γ , $d(P,\mathcal{L})$ is distance from point P to the line $\mathcal{L}$,and $d(P,P_0)$ is the distance from point P to the point P_0 **[27].**

Theorem 3.7

For every $\triangle$-regular curve Γ, there exists a tangent line to Γ at P_0 and the **tangent vector** is $\triangle$-differential of it's position vector function $r^{\triangle}(t_0)$, where $r(t_0) = P_0$ for $t_0 \in \mathbb{T}$.

Proof. This theorem can be proven as in [**27**], Theorem 3.3.

Definition 3.8

Let Γ be $\triangle$-regular curve and completely differentiable. The plane passing through points $P_0 \in \Gamma$ and orthogonal to the tangent vector to Γ at P_0 is called the **normal plane** to Γ at P_0, and let denoted by $\hat{a}$.

Since this plane is orthogonal to the vector $\Gamma^{\triangle}$ and contains the point with position vector $\hat{a} - \Gamma(t_0)$, then the equation of the normal plane is

$$(\hat{a} - \Gamma(t_0)) \cdot \Gamma^{\triangle}(t_0) = 0 \ \cdot$$

The vectors orthogonal to the tangent is called **normal vectors** to Γ [**25, 26**]**.**

Definition 3.9

Let Γ be $\triangle$-regular curve and completely differentiable. The plane with normal direction $\Gamma(P_0)$ and orthogonal the normal plane of Γ at P_0 is called **binormal plane,** the equation of the from

$\Gamma_1^{\triangle}(t_0)x + \Gamma_2^{\triangle}(t_0)y + \Gamma_3^{\triangle}(t_0)z = 0$, where x, y and z are Euclidean coordinate functions [**26**]**.**

Definition 3.10

Let P_0 be a point of $\triangle$-regular curve and take two points $Q_1, Q_2 \in \Gamma$ situated right side of P_0^{σ}. If the points Q_1 and Q_2 tend to P_0^{σ}, then the limit position of the plane containing P_0, P_0^{σ}, Q_1 and Q_2 is called the **osculating plane** of Γ at the point P_0 [**25**]**.**

Theorem 3.11

Let Γ be $\triangle$-regular curve and represented as $\Gamma = \Gamma(t)$. Assume that the vectors $\Gamma^{\triangle}$ and $\Gamma^{\triangle^2}$are not collinear at point P_0, then there exists osculating plane of Γ at P_0 and it is spanned by the vectors $\Gamma^{\triangle}$ and $\Gamma^{\triangle^2}$.

Proof. This theorem can be proven as in **[25]**, Theorem 4.1.

Theorem 3.12

The osculating plane of a planner curve coincides with the plane containing this curve.

Proof. This theorem can be proven as in **[25]**, Theorem 4.3.

4. FOCAL CURVE ON TIME SCALE

After we studied the previous sections, we mixed the curve with time scale that led me to find six new definitions and equations: $\triangle$-Tangent vector, $\triangle$-Curvature, $\triangle$-Normal vector, $\triangle$-Binormal vector, $\triangle$-Torsion and $\triangle$-Focal curve.

Definition 4.1. ($\triangle$-Tangent vector)

Let $\mathbb{T}$ be a time scale and $\Gamma : \mathbb{T} \to \mathbb{R}^3$ is $\triangle$-regular curve then the $\triangle$-tangent vector along Γ is defined by

$$T_{\triangle}(t) = \frac{\Gamma^{\triangle}(t)}{|\Gamma^{\triangle}(t)|} \ , \qquad \text{where } t \in \mathbb{T} \ .$$

Definition 4.2. ($\triangle$-Curvature)

Let $\mathbb{T}$ be a time scale and $\Gamma : \mathbb{T} \to \mathbb{R}^3$ is $\triangle$-regular curve then the $\triangle$-curvature is defined by

$$k_{\triangle} = \frac{\left|T_{\triangle}^{\triangle}(t)\right|}{|\Gamma^{\triangle}(t)|} \ , \qquad \text{where } t \in \mathbb{T} \ .$$

Definition 4.3. ($\triangle$-Normal vector)

Let $\mathbb{T}$ be a time scale and $\Gamma : \mathbb{T} \to \mathbb{R}^3$ is $\triangle$-regular curve then the $\triangle$-normal vector is defined by

$$N_{\triangle}(t) = \frac{T_{\triangle}^{\triangle}(t)}{k_{\triangle}} \ , \qquad \text{where } t \in \mathbb{T} \ .$$

Definition 4.4. ($\triangle$-Binormal vector)

Let $\mathbb{T}$ be a time scale and $\Gamma : \mathbb{T} \to \mathbb{R}^3$ is $\triangle$-regular curve then the $\triangle$-binormal vector along Γ is defined by

$$B_{\triangle}(t) = T_{\triangle}(t) \times N_{\triangle}(t) \ , \qquad \text{where } t \in \mathbb{T} \ .$$

Definition 4.5. ($\triangle$-Torsion)

Let $\mathbb{T}$ be a time scale and $\Gamma : \mathbb{T} \to \mathbb{R}^3$ is $\triangle$-regular curve then the $\triangle$-torsion of Γ is defined by

$$\tau_\triangle(t) = \frac{-B_\triangle^\triangle(t) \cdot N_\triangle(t)}{|\Gamma^\triangle(t)|} \ , \quad \text{where } t \in \mathbb{T} \ .$$

Definition 4.6. ($\triangle$-Focal curve)

Let $\mathbb{T}$ be a time scale and $\Gamma : \mathbb{T} \to \mathbb{R}^3$ is $\triangle$-regular curve then the $\triangle$-focal curve is defined by

$$C_\Gamma(t) = \Gamma(t) + c_\triangle N_\triangle(t) + e_\triangle B_\triangle(t) \ ,$$

where $c_\triangle = \dfrac{1}{k_\triangle}$ is first $\triangle$-focal curvature, $e_\triangle = \dfrac{c_\triangle^\triangle}{\tau_\triangle}$ is second $\triangle$-focal curvature and $t \in \mathbb{T}$.

Remark 4.7:

If $\tau_\triangle = 0$ then $\triangle$-focal curve is a planner.

Example 4.8.

If $\mathbb{T}$ be a time scale and $\Gamma : \mathbb{T} \to \mathbb{R}^3$ be $\triangle$-regular curve then find $\triangle$-focal curve for $\Gamma(t) = (\cos_p(t, t_0), \sin_p(t, t_0), 1)$ when $\mathbb{T} = \mathbb{R}$ and $\mathbb{T} = \mathbb{Z}$.

Solution:

If $\mathbb{T} = \mathbb{R}$ then $p = 1$ and $t_0 = 0$ so $\Gamma(t) = (\cos t, \sin t, 1)$

so $\Gamma^\triangle(t) = \Gamma'(t) = (-\sin t, \cos t, 0)$

$$\left|\Gamma'(t)\right| = \sqrt{\sin^2 t + \cos^2 t} = 1$$

$$T_\triangle(t) = \frac{\Gamma^\triangle(t)}{|\Gamma^\triangle(t)|} = \frac{\Gamma'(t)}{|\Gamma'(t)|} = \frac{(-\sin t, \cos t, 0)}{1} = (-\sin t, \cos t, 0)$$

$$T_\triangle^\triangle(t) = T_\triangle'(t) = (-\cos t, -\sin t, 0)$$

$$\left|T_{\triangle}^{\triangle}(t)\right| = \left|T_{\triangle}'(t)\right| = \sqrt{\cos^2 t + \sin^2 t} = 1$$

$$k_{\triangle} = \frac{\left|T_{\triangle}^{\triangle}(t)\right|}{\left|\Gamma^{\triangle}(t)\right|} = \frac{\left|T_{\triangle}'(t)\right|}{\left|\Gamma'(t)\right|} = 1$$

$$N_{\triangle}(t) = \frac{T_{\triangle}^{\triangle}(t)}{k_{\triangle}} = \frac{T_{\triangle}'(t)}{k_{\triangle}} = (-\cos t, -\sin t, 0)$$

$$B_{\triangle}(t) = T_{\triangle}(t) \times N_{\triangle}(t) = \begin{vmatrix} e_1 & e_2 & e_3 \\ -\sin t & \cos t & 0 \\ -\cos t & -\sin t & 0 \end{vmatrix} = (0,0,1)$$

$$B_{\triangle}^{\triangle}(t) = B_{\triangle}'(t) = (0,0,0)$$

$$\tau_{\triangle}(t) = \frac{-B_{\triangle}^{\triangle}(t) \cdot N_{\triangle}(t)}{\left|\Gamma^{\triangle}(t)\right|} = \frac{-B_{\triangle}'(t) \cdot N_{\triangle}(t)}{\left|\Gamma^{\triangle}(t)\right|} = \frac{-(0,0,0) \cdot (-\cos t, -\sin t, 0)}{1} = 0$$

then $\triangle$-focal curve is a planner.

$c_{\triangle} = \dfrac{1}{k_{\triangle}} = 1$ is first $\triangle$-focal curvature

so we can find $\triangle$-focal curve by

$$\begin{aligned} C_{\Gamma}(t) &= (\Gamma + c_{\triangle} N_{\triangle})(t) \\ &= ((\cos t, \sin t, 1) + (-\cos t, -\sin t, 0))(t) \\ &= (0,0,1)(t) \\ &= (0,0,t) \end{aligned}$$

If $\mathbb{T} = \mathbb{Z}$ then $p = 1$ and $t_0 = 0$ so $\Gamma(t) = (\cos t, \sin t, 1)$

$$\begin{aligned} \Gamma^{\triangle}(t) &= \triangle\Gamma(t) = \Gamma(t+1) - \Gamma(t) \\ &= (\cos(t+1), \sin(t+1), 1) - (\cos t, \sin t, 1) \\ &= (\cos(t+1) - \cos t, \sin(t+1) - \sin t, 0) \end{aligned}$$

$$\begin{aligned} \left|\Gamma^{\triangle}(t)\right| &= \left|\triangle\Gamma(t)\right| \\ &= \sqrt{\cos^2(t+1) - 2\cos(t+1)\cos t + \cos^2(t) + \sin^2(t+1) - 2\sin(t+1)\sin t + \sin^2 t} \\ &= \sqrt{1 + 1 - 2(\cos(t+1)\cos t + \sin(t+1)\sin t} \\ &= \sqrt{2 - 2\cos(t+1-t)} = \sqrt{2 - 2\cos(1)} = 0.958\,85 \end{aligned}$$

$$T_{\triangle}(t) = \frac{\Gamma^{\triangle}(t)}{\left|\Gamma^{\triangle}(t)\right|} = \frac{\triangle\Gamma(t)}{\left|\triangle\Gamma(t)\right|} = \frac{(\cos(t+1) - \cos t, \sin(t+1) - \sin t, 0)}{0.958\,85}$$

$$T_{\triangle}^{\triangle}(t) = \triangle T_{\triangle}(t) = T_{\triangle}(t+1) - T_{\triangle}(t)$$

$$= \frac{(\cos(t+2)-\cos(t+1), \sin(t+2)-\sin(t+1), 0)}{0.958\,85} - \frac{(\cos(t+1)-\cos t, \sin(t+1)-\sin t, 0)}{0.958\,85}$$

$$= \frac{(\cos(t+2)-2\cos(t+1)+\cos t, \sin(t+2)-2\sin(t+1)+\sin t, 0)}{0.958\,85}$$

$$\left|T_{\triangle}^{\triangle}(t)\right| = |\triangle T_{\triangle}(t)| = \sqrt{\frac{\begin{gathered}(\cos(t+2)-2\cos(t+1)+\cos t)^2+\\(\sin(t+2)-2\sin(t+1)+\sin t)^2\end{gathered}}{(0.958\,85)^2}}$$

$$= \sqrt{\frac{\begin{gathered}\cos^2(t+2)+4\cos^2(t+1)+\cos^2 t-4\cos(t+2)\cos(t+1)+\\2\cos(t+2)\cos t-4\cos(t+1)\cos t+\sin^2(t+2)+4\sin^2(t+1)+\\\sin^2 t-4\sin(t+2)\sin(t+1)+2\sin(t+2)\sin t-4\sin(t+1)\sin t\end{gathered}}{(0.958\,85)^2}}$$

$$= \sqrt{\frac{1+4+1-4\cos(t+2-t-1)+2\cos(t+2-t)-4\cos(t+1-t)}{(0.958\,85)^2}}$$

$$= \sqrt{\frac{6-4\cos(1)+2\cos(2)-4\cos(1)}{(0.958\,85)^2}}$$

$$= \sqrt{\frac{0.845\,29}{(0.958\,85)^2}} = 0.958\,85$$

$$k_{\triangle} = \frac{|\triangle T_{\triangle}(t)|}{|\triangle\Gamma(t)|} = \frac{0.958\,85}{0.958\,85} = 1$$

$$N_{\triangle}(t) = \frac{\triangle T(t)}{k_{\triangle}} = \frac{\frac{(\cos(t+2)-2\cos(t+1)+\cos(t), \sin(t+2)-2\sin(t+1)+\sin t, 0)}{0.958\,85}}{1}$$

$$= \frac{(\cos(t+2)-2\cos(t+1)+\cos(t), \sin(t+2)-2\sin(t+1)+\sin t, 0)}{0.958\,85}$$

$$B_{\triangle}(t) = T_{\triangle}(t) \times N_{\triangle}(t) = \begin{vmatrix} e_1 & e_2 & e_3 \\ \frac{\cos(t+1)-\cos t}{0.958\,85} & \frac{\sin(t+1)-\sin t}{0.958\,85} & 0 \\ \frac{\cos(t+2)-2\cos(t+1)+\cos t}{0.958\,85} & \frac{\sin(t+2)-2\sin(t+1)+\sin t}{0.958\,85} & 0 \end{vmatrix}$$

$$= (0, 0, \frac{(\cos(t+1)-\cos t)\cdot(\sin(t+2)-2\sin(t+1)+\sin t)}{0.958\,85} - \frac{(\sin(t+1)-\sin t)\cdot(\cos(t+2)-2\cos(t+1)+\cos t)}{0.958\,85})$$

$$
\begin{aligned}
&= (0,0,\frac{\begin{array}{c}\cos(t+1)\sin(t+2)-2\cos(t+1)\sin(t+1)+\cos(t+1)\sin t\\ -\sin(t+2)\cos t+2\sin(t+1)\cos t-\cos t\sin t\\ -\sin(t+1)\cos(t+2)+2\sin(t+1)\cos(t+1)-\sin(t+1)\cos t\\ +\cos(t+2)\sin t-2\cos(t+1)\sin t+\sin t\cos t\end{array}}{0.958\,85})\\
&= (0,0,\frac{\sin(t+2-t-1)+\sin(t-t-1)+\sin(t-t-2)+2\sin(t+1-t)}{0.958\,85})\\
&= (0,0,\frac{\sin(1)-\sin(1)-\sin(2)+2\sin(1)}{0.958\,85}) = (0,0,\frac{2\sin(1)-\sin(2)}{0.958\,85})\\
&= (0,0,0.806\,85)
\end{aligned}
$$

$$B_\triangle^\triangle(t) = \triangle B_\triangle(t) = (0,0,0.806\,85-0.806\,85) = (0,0,0)$$

$$\tau_\triangle = \frac{-B_\triangle^\triangle(t)\cdot N_\triangle(t)}{|\Gamma^\triangle(t)|} = \frac{-\triangle B_\triangle(t)\cdot N_\triangle(t)}{|\triangle\Gamma(t)|} = 0$$

so $\triangle$-focal curve is a planner

$c_\triangle = \dfrac{1}{k_\triangle} = 1$ is the first $\triangle$-focal curvature,

and we can find $\triangle$-focal curvature by

$$
\begin{aligned}
C_\Gamma(t) &= \Gamma(t) + c_\triangle N_\triangle(t)\\
&= ((\cos t,\sin t,1) + \frac{(\cos(t+2)-2\cos(t+1)+\cos(t),\sin(t+2)-2\sin(t+1)+\sin t,0)}{0.958\,85})(t)\\
&= (\frac{\cos(t+2)-2\cos(t+1)+2\cos t \quad , \quad \sin(t+2)-2\sin(t+1)+2\sin t \quad , \quad t}{0.958\,85}) \ .
\end{aligned}
$$

CONCLUSION

I have founded and created six new definitions (Δ-tangent vector, Δ-curvature, Δ-normal vector, Δ-binormal vector, Δ-torsion and Δ-focal curve) wich led to get six new equations that helped me to solve some geometrical examples on time scale, when $\mathbb{T}=\mathbb{R}$ and $\mathbb{T}=\mathbb{Z}$.

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