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ON THE DUALS OF SOME GENERALIZED SETS OF DIFFERENCE SEQUENCES

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Master Thesis
Department: Mathematics
Program: Analysis
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FOREWORD

First of all, I would like to thank God for giving me the strength and courage to complete my master thesis. And I am pleasured to show my great thank presentation to my supervisor "Professor Dr. Çiğdem BEKTAŞ" for her great efforts & valuabe time that she spent in supervising my thesis in all details and accurately. I also want to say thanks to my teacher "Dr.Sinan ERCAN" for helping me in preparing this project. Finally, I want to say thanks to everyone that helped me in preparing this thesis.

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SUMMARY

ON THE DUALS OF SOME GENERALIZED SETS OF DIFFERENCE SEQUENCES

In this thesis, we first mention the definitions of some fundamental notions and write some basic theorems and inequalities. Then we give some sequence sets $c_0(u, \triangle, p)$, $c(u, \triangle, p)$ and $\ell_\infty(u, \triangle, p)$ which are defined by Ç. Asma and R. Çolak [14], and also describe some properties of these sets. Moreover, we study on $\alpha-$ and $\beta-$ dual spaces of these sequence sets and give some theorems related by such dual spaces.

Keywords: Paranormed sequence space, $\alpha-$ and $\beta-$ dual spaces, difference sequence space.

ÖZET

BAZI GENELERTİRİLMİŞ FARK DİZİ UZAYLARI VE DUALLERİ

Bu tezde, ilkolarak bazı temel tanımları, teoremleri ve eşitsizlikleri verdik. Daha sonra Ç. Asma ve R. Çolak [14] tarafindan tanımlanan $c_0(u, \triangle, p)$, $c(u, \triangle, p)$ ve $\ell_\infty(u, \triangle, p)$ dizi uzaylarını ve bu uzayların bazı özelliklerini verdik. Dahası bu uzayların $\alpha-$ ve $\beta-$ duallerini verip bunlarla ilgili teoremler ve ispatlar verdik.

Anahtar kelimeler: Paranormlu dizi uzayları, $\alpha-$ ve $\beta-$ dual, fark dizi uzayları.

LIST OF SYMBOLS

i.e	:	It means,
iff	:	If and only if,
θ	:	Zero vector in a linear space,
$\mathbb{N}$	:	Set of all netural numbers,
$\mathbb{R}$	:	Set of all real numbers,
$\mathbb{R}^+$	:	Set of all non-negative real numbers,
$\mathbb{C}$	:	Set of all complex numbers,
ω	:	Set of all sequences with complex terms,
ℓ_∞	:	Space of bounded sequences,
c	:	Space of convergent sequences,
c_0	:	Space of null sequences,
$\triangle$	:	Difference operator.

1. INTRODUCTION

1.1 Fundamental Definitions and Theorems

Definition 1.1.1: By a vector space (linear space) we mean a non empty set X with two operations:

$$(x, y) \longrightarrow x + y \text{ from } X \times X \text{ into } X \text{ is said to be addition}$$

$$(\lambda, x) \longrightarrow \lambda.x \text{ from } K \times X \text{ into } X \text{ is said to be multiplication scalars}$$

such that the conditions below are satisfied for every elements x, y and z are in X and $\alpha, \beta \in K$:

(1) $x + y = y + x$,

(2) $(x + y) + z = x + (y + z)$,

(3) there exists an element $\theta \in X$, is said to be zero vector such that $x + \theta = x$,

(4) for each $x \in X$, there is an element $(-x) \in X$ is said to be the negative of x, such that $x + (-x) = \theta$,

(5) $\alpha(\beta x) = (\alpha\beta)x$,

(6) $(\alpha + \beta)x = \alpha x + \beta x$,

(7) $\alpha(x + y) = \alpha x + \alpha y$,

(8) $1.x = x$.

Elements of X are said to be vectors. If $K = \mathbb{R}$, then X is said to be a real linear space, and if $K = \mathbb{C}$, X is said to be a complex linear space [6, 7].

Definition 1.1.2: A subset M of a linear space X is said to be linear subspace in X if $\alpha x + \mu y \in M$ for all $x, y \in M$ and α, μ are scalars [2].

Definition 1.1.3: Let X be a non-empty set and $d : X \times X \longrightarrow \mathbb{R}^+$ be a distance function. Then (X, d) is said to be a metric space and d is a metric on X, if the following conditions are satisfied for every x, y and z are in X;

(M.1) $d(x, y) \geq 0$,

(M.2) $d(x, y) = 0$ if and only if $x = y$,

(M.3) $d(x, y) = d(y, x)$ (the symmetry property),

(M.4) $d(x, z) \leq d(x, y) + d(y, z)$ (the triangle inequality).

If the conditions (M.1), (M.3) and (M.4) are satisfied then (X, d) is called a semi-metric space and d is a semi-metric for X [3,7].

Remark 1.1.4:

(i) A metric (distance) function is a real valued function defined on arbitrary two elements in X.

(ii) The axioms (M.1), (M.2) and (M.3) for a metric space referred to as the Hausdorff postulates.

Definition 1.1.5: Let (X, d) be a metric space. Then d is called a bounded metric if for every $x, y \in X$, there is a constant $K > 0$ such that $d(x, y) \leq K$ [2].

Definition 1.1.6: A sequence (x_n) in a metric space (X, d) is a Cauchy sequence if for every $\varepsilon > 0$, there is an $N = N(\varepsilon)$ such that for every $m, n \geq N$, $d\ (x_n, x_m) < \varepsilon$ $[2, 4]$.

Definition 1.1.7: Let (x_n) be sequence in a metric space (X, d), then (x_n) is called convergent if there exists an $x \in X$ such that

$$\lim_{n \longrightarrow \infty} d\ (x_n, x) = 0,$$

x is said to be the limit of (x_n) and we write

$$\lim_{n \longrightarrow \infty} x_n = x$$

or, simply $x_n \longrightarrow x$ [4].

Theorem 1.1.8: The following statements are hold:

(a) If a sequence (x_n) is convergent then it has a unique limit.

(b) If $\lim_{n \longrightarrow \infty} x_n = x$, then every subsequence of (x_n) converges to x.

(c) If a sequence (x_n) in a metric space (X, d) is convergent, then it is a Cauchy sequence. But the converse is not generally true [5].

Theorem 1.1.9 (Bolzano-Weierstrass): Every bounded sequence has a convergent subsequence.

Definition 1.1.10: A metric space $(X, d\ )$ is complete if every Cauchy sequence in X converges to an element of X [5].

Definition 1.1.11: Let X be a real or complex linear space and d be a metric for X. Then $(X, d\ )$ is called a linear metric space, if the algebraic operations are continuous on X. A Frechet space is defined to be a complete linear metric space $[3,13]$.

Definition 1.1.12: Let X be a linear space over the field $\mathbb{R}$ or $\mathbb{C}$ and $\|\cdot\| : X \longrightarrow \mathbb{R}^+$ be a function, then we say that $(X, \|\cdot\|)$ is a normed linear space and $\|\cdot\|$ is a norm on X, if the norm axioms below are satisfied for every x, y in X and for every scalar λ:

(N.1) $\|x\| \geq 0$,

(N.2) $\|x\| = 0$ if and only if $x = \theta$,

(N.3) $\|\lambda x\| = |\lambda| . \|x\|$ (absolute homogeneity),

(N.4) $\|x + y\| \leq \|x\| + \|y\|$ (triangle inequality).

If X is a real linear space, then λ must be a real number [3].

Theorem 1.1.13: Every normed space is a metric space. It can be shown by using $\rho(x, y) = \|x - y\|$ [3] .

Definition 1.1.14: A Banach space X is defined to be a complete normed linear space [1].

Definition 1.1.15: Let X be a linear space over the field $\mathbb{R}$ or $\mathbb{C}$ and g from X into $\mathbb{R}$ be a function. Then (X, g) is said to be a paranormed space and g is a paranorm for X, if the axioms below are satisfied for every x,$y \in X$ and for every scalar α:

(g.1) $\mathrm{g}(x) = 0$ if $x = \theta$,

(g.2) $\mathrm{g}(-x) = \mathrm{g}(x)$,

(g.3) $\mathrm{g}(x + y) \leq \mathrm{g}(x) + \mathrm{g}(y)$,

(g.4) If (α_n) is a sequence of scalars with α_n coverges to α as $(n \longrightarrow \infty)$ and x_n,$x \in X$ for all $n \in \mathbb{N}$ with x_n converges to x then $\alpha_n.x_n$ converges to $\alpha.x$ as $(n \longrightarrow \infty)$, in the sense that $\mathrm{g}(\alpha_n.x_n - \alpha.x)$ converges to 0 as $(n \longrightarrow \infty)$.

A total paranormed space is a paranormed which satisfies this axiom $\mathrm{g}(x) = 0 \Longrightarrow x = 0$ $[3, 7]$.

Theorem 1.1.16:

(a) Every paranormed space is a linear semi metric space.

(b) Every total paranormed space is a linear metric space [7].

Definition 1.1.17: Let X be a normed space and (e_n) be a sequence of elements of X, then (e_n) is said to be a Shauder basis for X if for each element x of X there exists a unique sequence (λ_n) of scalars such that

$$\left\| x - \sum_{k=1}^{n} \lambda_k \, e_k \right\| \longrightarrow 0 \text{ as } n \longrightarrow \infty.$$

The series

$$\sum_{k=1}^{\infty} \lambda_k \, e_k$$

which corresponds the sum of x is said to be the expansion of x according to (e_n), and we write

$$x = \sum_{k=1}^{\infty} \lambda_k \, e_k. \quad [2,4].$$

Definition 1.1.18: Let X and Y be two normed linear spaces over the same field of scalars, then

(a) A transformation (or map, operator) $T : X \longrightarrow Y$ is said to be linear if

$$T(x+y) = T(x) + T(y) \text{ and } T(\lambda x) = \lambda T(x)$$

for all $x, y \in X$ and for all scalars λ.

(b) A linear transformation $T : X \longrightarrow Y$ is said to be continuous at $x \in X$ if

$$x_n \longrightarrow x \implies T(x_n) \longrightarrow T(x) \quad \text{as } n \longrightarrow \infty.$$

T is said to be continuous if T is continuous at each point of X.

(c) A linear transformation $T : X \longrightarrow Y$ is said to be bounded if there is a real number $K > 0$ such that

$$\|T\ (x)\| \leq K\ \|x\| \quad \forall\, x \in X \quad [2].$$

Definition 1.1.19: f is a linear functional on X if $f : X \longrightarrow \mathbb{R}$ or $\mathbb{C}$ is a linear operator. In other words a linear functional is a real or complex-valued linear operator [1].

Definition 1.1.20: Let X and Y be two normed spaces. The set $B(X,Y)$ consisting of all bounded linear operators from X into Y is a linear space with respect to the addition and scalar multiplication of operators. If $Y = \mathbb{R}$ or $\mathbb{C}$, then $B(X,Y)$ is called the dual space of X and denoted by X^*. X^* forms a normed space with the norm

$$\|f\| = \sup_{x\in X-\{\theta\}} \left\{ \frac{|f\ (x)|}{\|x\|} \right\} = \sup_{\substack{x\in X \\ \|x\|=1}} \{|f\ (x)|\}\ [4,1].$$

Definition 1.1.21: By ω, we define the set of all complex sequences , i.e.,

$$\omega = \{z = (z_k) : z_k \in \mathbb{C}\ ,\ (k \in \mathbb{N})\}\,.$$

Each element $z = (z_k)$ of ω is a sequence of the form $z = (x_k + iy_k)_{k=1}^{\infty}$, where (x_k) and (y_k) are both real sequences and $i = \sqrt{-1}$. It is a simple verication that ω is a linear space according to the usual co-ordinatewise addition and scalar multiplication of sequences which are defined by

$$x + y = (x_k) + (y_k) = (x_k + y_k) \quad \text{and} \quad \alpha x = \alpha\,(x_k) = (\alpha x_k)$$

respectively; where $x = (x_k)$, $y = (y_k) \in \omega$ and $\alpha \in \mathbb{C}$. The popular metric on ω is defined by

$$d_\omega(x,y) = \sum_{n=1}^{\infty} \frac{1}{2^n} \frac{|x_n - y_n|}{1 + |x_n - y_n|}; \quad x = (x_k),\ y = (y_k) \in \omega\ [3].$$

Definition 1.1.22:

(a) The space of all bounded sequences is denoted by ℓ_∞ and given as :

$$\ell_\infty = \left\{ x = (x_k) \in \omega\ :\ \sup_{k\in\mathbb{N}} |x_k| < \infty \right\}\ [3].$$

(b) The space of all convergent sequences is denoted by c and given as :

$$c = \left\{ x = (x_k) \in \omega\ :\ \lim_{k\longrightarrow\infty} |x_k - l\ | = 0\ ,\ \text{for some } l \in \mathbb{C} \right\} [3].$$

(c) The space of all null sequences is denoted by c_0 and given as :

$$c_0 = \left\{ x = (x_k) \in \omega\ :\ \lim_{k\longrightarrow\infty} x_k = 0 \right\}\ [3].$$

The natural metric d on λ, $\lambda \in \{\ell_\infty,\ c,\ c_0\}$ is defined by

$$d_\infty(x,y) = \sup_{n \in \mathbb{N}} |x_n - y_n|$$

where $x = (x_n)$ and $y = (y_n)$ are in λ [7].

Definition 1.1.23: The space of all sequences (x_k) such that $\sum_{k=1}^{\infty} |x_k|^p$ converges where $0 < p < \infty$ is denoted by ℓ_p and given as :

$$\ell_p = \left\{ x = (x_k) \in \omega \ : \ \sum_{k=1}^{\infty} |x_k|^P \ \text{ converges} \right\} \ [3].$$

The natural metrics d_p and $\widetilde{d_p}$ on ℓ_p are given by

$$\begin{aligned} d_p(x,y) &= \left(\sum_{k=1}^{\infty} |x_k - y_k|^p \right)^{\frac{1}{p}} \quad , \quad p \geq 1 \\ \widetilde{d_p}(x,y) &= \sum_{k=1}^{\infty} |x_k - y_k|^p \quad , \ 0 < p < 1 \end{aligned}$$

where $x = (x_k)$ and $y = (y_k)$ are in ℓ_p. Note that if we take $p = 1$ then we get ℓ_1 [1].

Definition 1.1.24: The spaces bs, cs and cs_0 of bounded series, convergent series and the series converging to zero respectively, are defined by

$$\begin{aligned} bs &= \left\{ x = (x_k) \in \omega : \sup_{n \in \mathbb{N}} \left| \sum_{k=0}^{n} x_k \right| < \infty \right\}, \\ cs &= \left\{ x = (x_k) \in \omega : \lim_{k \to \infty} \left| \sum_{k=0}^{n} x_k - \ell \right| = 0 \ \text{ for some } \ell \in \mathbb{C} \right\}, \\ cs_0 &= \left\{ x = (x_k) \in \omega : \lim_{k \to \infty} \left| \sum_{k=0}^{n} x_k \right| = 0 \right\} \ [3]. \end{aligned}$$

The natural metric on the spaces bs, cs and cs_0 is defined by

$$d(x,y) = \sup_{n \in \mathbb{N}} \left| \sum_{k=0}^{n} (x_k - y_k) \right|.$$

1.2 Some Inequalities

(i) Triangle inequality: For any $a\ ,b \in \mathbb{C}$, then

$$|a+b| \leq |a| + |b| \quad [1]\,. \tag{1.1}$$

(ii) Let $a, b \in \mathbb{C}$ and $0 < p \leq 1$. Then, we have

$$|a+b|^p \leq |a|^p + |b|^p \quad [7]\,. \tag{1.2}$$

(iv) Minkowski's inequality: Let $1 \leq p < \infty$, then

$$\left[\sum_{i\ =1}^{\infty} |x_i + y_i|^p\right]^{\frac{1}{p}} \leq \left[\sum_{i\ =1}^{\infty} |x_i|^p\right]^{\frac{1}{p}} + \left[\sum_{i\ =1}^{\infty} |y_i|^p\right]^{\frac{1}{p}}, \tag{1.3}$$

where $x = (x_i)$, $y = (\ y_i) \in \ell_p$ and $i = 1, 2, ...$ [3].

2. SOME SEQUENCE SPACES

Suppose here and after that $p=(p_k)$ is a sequence of strictly positive real numbers and let $M = \max\{1, \sup_k p_k\}$. We also assume that U is the set of all non zero sequences $u=(u_k)_{k=1}^{\infty}$ with complex terms. We shall use these throughout the next.

Definition 2.1: The spaces $\ell_\infty(p)$, $c(p)$ and $c_0(p)$ were introduced and investigated by various authors and defined by Maddox [8] (see also [9],[3]) as follows:

$$\begin{aligned}
\ell_\infty(p) &= \left\{x=(x_k)\in\omega:\ \sup_{k\in\mathbb{N}}\ |x_k|^{p_k}<\infty\right\}, \\
c(p) &= \left\{x=(x_k)\in\omega:\ \lim_{k\longrightarrow\infty}|x_k-\ell\,|^{p_k}=0,\ \text{for some } \ell\in\mathbb{C}\right\}, \\
c_0(p) &= \left\{x=(x_k)\in\omega:\ \lim_{k\longrightarrow\infty}|x_k|^{p_k}=0\right\}.
\end{aligned}$$

It's easy to see that each of the sets $\ell_\infty(p)$, $c(p)$ and $c_0(p)$ forms a linear space with respect to the usual co-ordinatewise operations.

Note 2.2: If the terms of (p_k) are constants and all equal to $p>0$, then we see that $\ell_\infty(p)=\ell_\infty$, $c_0(p)=c_0$, and $c(p)=c$.

Theorem 2.3: $c_0(p)$ is a complete and linear metric space paranormed by

$$\mathrm{g}(x)=\sup_{k\in\mathbb{N}}\ |x_k|^{\frac{p_k}{M}}\ . \tag{2.1}$$

If $\inf p_k > 0$ then $\ell_\infty(p)$ and $c(p)$ are also complete and linear metric spaces paranormed by (2.1) [7].

Proof: It's known that a total paranormed space implies a linear metric space. We here only consider the proof of the continuity of scalar multiplication on $c_0(p)$. Now let $x=(x_k)$ be an arbitrary sequence in $c_0(p)$ and λ be any scalar in $\mathbb{C}$. Since the inequality

$$|\lambda|^{p_k}\le\max\left\{1,|\lambda|^{M}\right\}$$

holds for $\lambda\in\mathbb{C}$, we have

$$|\lambda|^{\frac{p_k}{M}}\le\left[\max\left\{1,|\lambda|^{M}\right\}\right]^{\frac{1}{M}}=\max\{1,|\lambda|\}\,. \tag{2.2}$$

Hence

$$g(\lambda x) \leq \max\{1, |\lambda|\}\ g(x).$$

Thus the function $(\lambda, x) \longrightarrow \lambda x$ is continuous at $\lambda = 0$, $x = 0$ and the function $x \longrightarrow \lambda x$ is continuous at $x = 0$ when λ is fixed. Now let x be fixed in $c_0(p)$ and $\varepsilon > 0$ then we choose $N \in \mathbb{N}$ such that

$$\sup_{k > N} |x_k|^{\frac{p_k}{M}} < \frac{\varepsilon}{2}$$

for $\delta > 0$, so that $|\lambda| < \delta$ gives

$$\sup_{k \leq N} |\lambda x_k|^{\frac{p_k}{M}} < \frac{\varepsilon}{2}.$$

Thus if $|\lambda| < \min\{1, \delta\}$, then

$$\begin{aligned} g(\lambda x) &= \sup_{k \leq N} |\lambda x_k|^{\frac{p_k}{M}} + \sup_{k > N} |x_k|^{\frac{p_k}{M}} \\ &< \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon. \end{aligned}$$

Thus the function $\lambda \longrightarrow \lambda x$ is continuous at $\lambda = 0$. Hence we get the proof. The completness of above spaces is known and the reader may see it in [7] for details. To show that the second part of above theorem is fail only in which continuity of scalar multiplication but not in special case, we here give an example.

Let $(p_m) = (\frac{1}{m})$ and $(x_m) = (1, 1, ...)$ for every m, then $\ell_\infty(p)$ contains x. Let λ be such as $0 < \lambda < 1$, then $|\lambda|^{\frac{1}{m}} < 1$ for all m and $|\lambda|^{\frac{1}{m}} \longrightarrow 1$ as $m \longrightarrow \infty$, so that

$$g(\lambda x) = \sup_m |\lambda x_m|^{\frac{1}{m}} = \sup_m |\lambda|^{\frac{1}{m}} = 1.$$

Hence $\lambda x \nrightarrow 0$ as $\lambda \longrightarrow 0$ and this clarifies that $\ell_\infty(p)$ is not a linear metric space in general case.

Note that by Definition 1.1.11, the space $c_0(p)$ forms a Frechet space, the other two spaces $\ell_\infty(p)$ and $c(p)$ are also forming Frechet spaces with considering their special cases.

Lemma 2.4: The following inclusion relations are strictly hold.

(i) $c_0(p) \subset c(p)$,

(ii) $c(p) \subset \ell_\infty(p)$.

Proof : The proof is trivial, so we omit it.

Note that the inclusions are strict, for instance in the first part of above inclusions if we take $p_k = 1$, $x_k = \frac{1}{k} + 1$ for every k, then $c(p)$ contains x_k and $\lim_{k \longrightarrow \infty} |x_k|^{p_k} = \lim_{k \longrightarrow \infty} \left| \frac{1}{k} + 1 \right| = 1$, but $x_k \notin c_0(p)$.

Definition 2.5: The following sequence sets has been defined by Kizmaz [16] as

$$\begin{aligned} \ell_\infty(\triangle) &= \{x = (x_k) : \triangle x \in \ell_\infty\}, \\ c(\triangle) &= \{x = (x_k) : \triangle x \in c\}, \\ c_0(\triangle) &= \{x = (x_k) : \triangle x \in c_0\}, \end{aligned}$$

where $\triangle x = (\triangle x_k) = (x_k - x_{k+1})$.

Theorem 2.6: The sequence sets $\ell_\infty(\triangle), c(\triangle)$ and $c_0(\triangle)$ are Banach spaces with the following norm

$$\|x\| = |x_1| + \sup_k |x_k| .$$

Proof : To see the proof entirely refer to [16].

Lemma 2.7: $\sup_k |\triangle x_k| = \sup_k |x_k - x_{k+1}| < \infty$ if and only if

(i) $\sup_k k^{-1} |x_k| < \infty$ and (ii) $\sup_k |x_k - k(k+1)^{-1} x_{k+1}| < \infty$.

Definition 2.8: The following sequence sets are defined as;

$$\begin{aligned} \ell_\infty(u,p) &= \left\{ x = (x_k) \in \omega : \sup_k |u_k x_k|^{p_k} < \infty \right\}, \\ c_0(u,p) &= \left\{ x = (x_k) \in \omega : \lim_{k \longrightarrow \infty} |u_k x_k|^{p_k} = 0 \right\}, \\ c(u,p) &= \left\{ x = (x_k) \in \omega : \lim_{k \longrightarrow \infty} |u_k x_k - \ell\,|^{p_k} = 0, \text{ for some } \ell \in \mathbb{C} \right\} \text{ [18].} \end{aligned}$$

Theorem 2.9: The above sequence sets are paranormed by

$$\mathrm{g}(x) = \sup_k |u_k x_k|^{\frac{p_k}{M}} \text{ [18].}$$

Proof: The axioms (g.1) and (g.2) are trivial to satisfy so we here consider the other axioms. Let $x = (x_k)$ and $y = (y_k) \in c(u,p)$, then (g.3) follows from (1.2) as below

$$\begin{aligned} \mathrm{g}(x+y) &= \sup_k |u_k (x_k+y_k)|^{\frac{p_k}{M}} \\ &= \sup_k |u_k x_k + u_k y_k|^{\frac{p_k}{M}} \\ &\leq \sup_k |u_k x_k|^{\frac{p_k}{M}} + \sup_k |u_k y_k|^{\frac{p_k}{M}} \\ &= \mathrm{g}(x)+\mathrm{g}(y). \end{aligned}$$

Let λ be a scalar in $\mathbb{C}$, then (g.4) is proved as the same manner of the proof of Theorem 2.3.

Definition 2.10: The following sequence spaces have been defined by Asma and Çolak [14] as;

$$\begin{aligned} \ell_\infty(u,\triangle,p) &= \{x=(x_k)\in\omega : \ (u_k \triangle x_k)\in \ell_\infty(p)\}, \\ c(u,\triangle,p) &= \{x=(x_k)\in\omega : \ (u_k \triangle x_k)\in c(p)\}, \\ c_0(u,\triangle,p) &= \{x=(x_k)\in\omega : \ (u_k \triangle x_k)\in c_0(p)\}, \end{aligned}$$

where $\triangle^0 x = (x_k)$, $\triangle x = (\triangle x_k) = (x_k - x_{k+1})$ for $k = 1, 2, ...$ and $u \in U$.

It's clear to see that if $p = (p_k)$ is a fixed real sequence such that $0 < p_k \leq \sup_k p_k < \infty$ and u is in U, then $c_0(u,\triangle,p)$, $c(u,\triangle,p)$ and $\ell_\infty(u,\triangle,p)$ are linear spaces, with respect to the usual co-ordinatewise operations

$$x+y=(x_n+y_n) \quad \text{and} \quad \mu x=(\mu x_n)$$

where μ represents an arbitrary number in $\mathbb{C}$. Note that if we here take $p_k = 1$ for all k in $\mathbb{N}$, we have $c_0(u,\triangle,p) = c_0(u,\triangle)$, $c(u,\triangle,p) = c(u,\triangle)$ and $\ell_\infty(u,\triangle,p) = \ell_\infty(u,\triangle)$, where

$$\begin{aligned} c_0(u,\triangle) &= \{x=(x_k)\in\omega : (u_k \triangle x_k)\in c_0\}, \\ c(u,\triangle) &= \{x=(x_k)\in\omega : (u_k \triangle x_k)\in c\}, \\ \ell_\infty(u,\triangle) &= \{x=(x_k)\in\omega : (u_k \triangle x_k)\in \ell_\infty\}. \end{aligned}$$

Theorem 2.11: Let $p = (p_k)$ be a sequence of strictly reals such that $0 < p_k \leq \sup_k p_k < \infty$ and $u \in U$. Then $c_0(u,\triangle,p)$ forms a paranormed space with the paranorm below

$$\mathrm{g}(x) = \sup_k |u_k \triangle x_k|^{\frac{p_k}{M}}. \tag{2.3}$$

If $\inf_k p_k > 0$ then the sets $\ell_\infty(u, \triangle, p)$ and $c(u, \triangle, p)$ form paranormed spaces with the same paranorm as (2.3) [14].

Proof: To prove the first part we follow the paranormed axioms by using the given information from above. Let $u \in U$ and $p = (p_k)$ be such above with $x = (x_k)$ and $y = (y_k)$ be in $c_0(u, \triangle, p)$.

(g.1) Now let $x = \theta = (0, 0, ...)$, then we have

$$\mathrm{g}(x) = \sup_k |u_k \triangle x_k|^{\frac{p_k}{M}} = \sup_k |u_k.\theta|^{\frac{p_k}{M}} = \sup_k |\theta|^{\frac{p_k}{M}} = 0.$$

(g.2)

$$\begin{aligned} \mathrm{g}(-x) &= \sup_k |u_k \triangle (-x_k)|^{\frac{p_k}{M}} \\ &= \sup_k |-u_k \triangle x_k|^{\frac{p_k}{M}} \\ &= \sup_k |u_k \triangle x_k|^{\frac{p_k}{M}} \\ &= \mathrm{g}(x). \end{aligned}$$

(g.3) By (1.2), we have

$$\begin{aligned} \mathrm{g}(x+y) &= \sup_k |u_k(\triangle x_k + \triangle y_k)|^{\frac{p_k}{M}} \\ &= \sup_k |u_k \triangle x_k + u_k \triangle y_k)|^{\frac{p_k}{M}} \\ &\leq \sup_k |u_k \triangle x_k|^{\frac{p_k}{M}} + \sup_k |u_k \triangle y_k|^{\frac{p_k}{M}} \\ &= \mathrm{g}(x) + \mathrm{g}(y). \end{aligned}$$

(g.4) Let $\lambda \in \mathbb{C}$ be a scalar then by (2.2), we have

$$\mathrm{g}(\lambda x) \leq \max\{1, |\lambda|\} \ \mathrm{g}(x).$$

so, $(\lambda, x) \longrightarrow \lambda x$ is continuous at $\lambda = 0$, $x = 0$ and so if λ is fixed then $x \longrightarrow \lambda x$ is continuous at $x = 0$. Now suppose x be a fixed in $c_0(u, \triangle, p)$ and $\varepsilon > 0$ be given, then we can choose $N \in \mathbb{N}$ such that

$$\sup_{k > N} |u_k \triangle x_k|^{\frac{p_k}{M}} < \frac{\varepsilon}{2}$$

for $\delta > 0$, so if $|\lambda| < \delta$ then it gives

$$\sup_{k \leq N} |\lambda\ u_k \bigtriangleup x_k|^{\frac{p_k}{M}} < \frac{\varepsilon}{2}.$$

Thus if $|\lambda| < \min\{1, \delta\}$, then

$$\begin{aligned} \mathrm{g}(\lambda x) &= \sup_{k \leq N} |\lambda\ u_k \bigtriangleup x_k|^{\frac{p_k}{M}} + \sup_{k > N} |u_k \bigtriangleup x_k|^{\frac{p_k}{M}} \\ &< \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon. \end{aligned}$$

Therefore $\lambda \longrightarrow \lambda x$ is continuous at $x = 0$. So we get the proof.

Note 2.12:

(a) Notice that in general case the second part of above theorem, i.e., $\ell_\infty(u, \bigtriangleup, p)$ and $c(u, \bigtriangleup, p)$ is fail in which continuity of scalar multiplication but not in special case. To clarify that we here give an example:

Let $(x_i) = (-2i)$, $(u_i) = \left(\frac{1}{2}\right)$ and $(p_i) = \left(\frac{1}{i^2}\right)$ for all i . Let $0 < |\alpha| < 1$. Then $|\alpha|^{\frac{1}{i^2}} < 1$ for all i and $|\alpha|^{\frac{1}{i^2}} \longrightarrow 1$ as $i \longrightarrow \infty$, so that

$$\mathrm{g}(\alpha x) = \sup_i |\alpha u_i \bigtriangleup x_i|^{\frac{p_i}{M}} = \sup_i \left|\alpha \frac{1}{2}\,(2)\right|^{\frac{1}{i^2}} = 1.$$

Hence $\alpha x \nrightarrow 0$ as $\alpha \longrightarrow 0$ and this clarifies that $\ell_\infty(u, \bigtriangleup, p)$ is not a paranormed space.

(b) Notice that the above paranormed spaces, i.e., $\ell_\infty(u, \bigtriangleup, p)$, $c(u, \bigtriangleup, p)$ and $c_0(u, \bigtriangleup, p)$, are not total, because in each case of them if

$$\mathrm{g}(x) = \sup_i |u_i \bigtriangleup x_i|^{\frac{p_i}{M}} = \theta, \text{ where } \theta = (0, 0, ...)\,,$$

does not imply $x = \theta$. For instance if $x_i = 1$, for every i then $x_i \in \ell_\infty(u, \bigtriangleup, p)$ and

$$\mathrm{g}(x) = \sup_i |u_i \bigtriangleup x_i|^{\frac{p_i}{M}} = \sup_i |\theta|^{\frac{p_i}{M}} = \theta,$$

but $x_i \neq \theta$.

(c) It's known that paranormed spaces imply linear semi metric spaces, hence the space $c_0(u, \bigtriangleup, p)$ is a linear semi metric with (2.3) and the other two spaces $\ell_\infty(u, \bigtriangleup, p)$ and $c(u, \bigtriangleup, p)$ are also linear semi metric by (2.3) with considering their special cases.

Lemma 2.13: The following inclusion relations are strictly hold.

(i) $c_0(u, \triangle, p) \subset c(u, \triangle, p)$,

(ii) $c(u, \triangle, p) \subset \ell_\infty(u, \triangle, p)$.

Proof:

(i) Let $x \in c_0(u, \triangle, p)$, then

$$\lim_{k \longrightarrow \infty} |u_k \triangle x_k|^{p_k} = 0.$$

If we take $\ell = 0$, we may write

$$\lim_{k \longrightarrow \infty} |u_k \triangle x_k - 0|^{p_k} = \lim_{k \longrightarrow \infty} |u_k \triangle x_k - \ell|^{p_k} = 0.$$

For arbitrary $x = (x_k)$ in $c_0(u, \triangle, p)$, we get $x \in c(u, \triangle, p)$. So $c_0(u, \triangle, p) \subset c(u, \triangle, p)$.

(ii) Let $x \in c(u, \triangle, p)$ and $\ell \in \mathbb{C}$ with H, $M \in \mathbb{R}$ then by (1.2), we have

$$\begin{aligned} \sup_k |u_k \triangle x_k|^{p_k} &= \sup_k |u_k \triangle x_k - \ell + \ell|^{p_k} \\ &\leq \sup_k |u_k \triangle x_k - \ell|^{p_k} + |\ell|^{p_k} \\ &= H + M < \infty. \end{aligned}$$

So, for every x in $c(u, \triangle, p)$, we get $x \in \ell_\infty(u, \triangle, p)$. Hence $c(u, \triangle, p) \subset \ell_\infty(u, \triangle, p)$. These complete the proof. Note that the inclusions are strict. For instance in first inclusion if $u_k = 1$, $p_k = \frac{1}{k}$ and $x_k = k$. Since

$$\lim_{k \longrightarrow \infty} |u_k \triangle x_k|^{\frac{p_k}{M}} = \lim_{k \longrightarrow \infty} |-1|^{\frac{1}{k}} = 1$$

so, $x_k \in c(u, \triangle, p) - c_0(u, \triangle, p)$.

3. KOTHE-TOEPLITZ DUALS

In this chapter we study on $\alpha-$ and $\beta-$ duals of the sequence sets $c_0(u,\triangle,p)$, $c(u,\triangle,p)$ and $\ell_\infty(u,\triangle,p)$ which are given in the last chapter. Suppose that $w_k=\frac{1}{|u_k|}$ where $u_k \in U$, we use this throughout the next.

Definition 3.1: For any two sequence spaces λ and ξ, the set $S(\lambda,\xi)$ is defined by

$$S(\lambda,\xi)=\{z=(z_k)\in\omega : xz=(x_kz_k)\in\xi \quad \text{for all} \quad x=(x_k)\in\lambda\} \tag{3.1}$$

is said to be multipllier space of λ and ξ. With the above notation (3.1), the $\alpha-$ and $\beta-$ duals of a sequence space λ which are denoted by λ^α, λ^β respectively, are defined as

$$\lambda^\alpha=S(\lambda,\ell_1) \qquad \text{and} \qquad \lambda^\beta=S(\lambda,cs)$$

that is

$$\begin{aligned}\lambda^\alpha &= \left\{x=(x_k)\in\omega : \sum_{k=0}^{\infty}|x_ky_k|<\infty, \text{ for all} \quad y=(y_k)\in\lambda\right\},\\ \lambda^\beta &= \left\{x=(x_k)\in\omega : \sum_{k=0}^{\infty}x_ky_k \ \text{ converges, for all} \quad y=(y_k)\in\lambda\right\} \text{ [3, 14]}.\end{aligned}$$

Note 3.2: λ^α, λ^β are called Köthe-Toeplitz and generalized Köthe-Toeplitz duals respectively. It's known that $\lambda^\alpha\subset\lambda^\beta$ and if $\lambda\subset\xi$ then $\xi^\eta\subset\lambda^\eta$, where $\eta=\alpha$ or β. Furthermore we write $\lambda^{\alpha\alpha}=(\lambda^\alpha)^\alpha$ and $\lambda^{\beta\beta}=(\lambda^\beta)^\beta$ [3, 14, 19].

Theorem 3.3: For every sequence $p=(p_k)$ of strictly real numbers, we have

(i) $[c_0(p)]^\alpha=M_0(p)$,

(ii) $[\ell_\infty(p)]^\alpha=M_\infty(p)$,

(iii) $[c(p)]^\alpha=M(p)$,

where

$$M_0(p)=\bigcup_{N>1}\left\{a=(a_k):\sum_{k=1}^{\infty}|a_k|\,N^{-\frac{1}{p_k}}<\infty\right\},$$

$$M_\infty(p)=\bigcap_{N>1}\left\{a=(a_k):\sum_{k=1}^{\infty}|a_k|\,N^{\frac{1}{p_k}}<\infty\right\},$$

and

$$M(p)=M_0(p)\cap\left\{a=(a_k):\sum_{k=1}^{\infty}|a_k|<\infty\right\} \text{ [19]}.$$

Proof: **(i)** Let $a = (a_k) \in M_0(p)$ and $x = (x_k) \in c_0(p)$. Then

$$\sum_{k=1}^{\infty} |a_k|\, N^{-\frac{1}{p_k}} < \infty$$

for some $N > 1$ and

$$|x_k|^{p_k} < N^{-1} \implies |x_k| < N^{-\frac{1}{p_k}}$$

for some large k, where for k it follows that

$$|a_k x_k| = |a_k|\,|x_k| \le |a_k|\, N^{-\frac{1}{p_k}},$$

hence

$$\begin{aligned} \sum_{k=1}^{\infty} |a_k\ x_k| &= \sum_{k=1}^{\infty} |a_k|\ \ |x_k| \\ &\le \sum_{k=1}^{\infty} |a_k|\, N^{-\frac{1}{p_k}} < \infty, \end{aligned}$$

so $M_0(p) \subset [c_0(p)]^{\alpha}$. Since $[c_0(p)]^{\alpha} \subset [c_0(p)]^{\beta}$ then by Theorem 6 in [20] it follows that $[c_0(p)]^{\alpha} = M_0(p)$.

(ii) It's similar to the above proof of (i), to see more details refer to [15] and [19].

(iii) Let $a = (a_k) \in M(p)$ and $x = (x_k) \in c(p)$, $|x_k - \ell|^{p_k} \longrightarrow 0$ $(k \longrightarrow \infty)$. Then

$$\sum_{k=1}^{\infty} |a_k| < \infty$$

and since $x \in c(p)$ then $(x_k - \ell) \in c_0(p)$ and so

$$\sum_{k=1}^{\infty} |a_k(x_k - \ell)| < \infty.$$

Now by the inequality

$$\begin{aligned} |a_k x_k| &= |a_k x_k - a_k\ \ell + a_k\ \ell| \\ &\le |a_k\,(x_k - \ell)| + |\ell\ a_k|, \end{aligned}$$

we get that

$$\sum_{k=1}^{\infty} |a_k\ x_k| < \infty.$$

Thus $a \in [c(p)]^{\alpha}$. Since $c_0(p) \subset c(p)$ it drives that $[c(p)]^{\alpha} \subset [c_0(p)]^{\alpha}$. Let $a \in [c(p)]^{\alpha}$, since $e = (1, 1, 1, ...) \in c(p)$, it drives that

$$\sum_{k=1}^{\infty} |a_k| < \infty,$$

so that $a \in \ell_1$, and hence $a \in [c_0(p)]^{\alpha} \cap \ell_1 = M(p)$. So we get the proof.

Theorem 3.4: For every sequence $p = (p_k)$ of strictly real numbers, we have
(i) $[c_0(p)]^{\alpha\alpha} = E_0$,
(ii) $[\ell_\infty(p)]^{\alpha\alpha} = E_\infty$,
where

$$E_0 = \bigcap_{N>1} \left\{ x = (x_k) : \sup_k |x_k| \, N^{\frac{1}{p_k}} < \infty \right\}$$

and

$$E_\infty = \bigcup_{N>1} \left\{ x = (x_k) : \sup_k |x_k| \, N^{-\frac{1}{p_k}} < \infty \right\} \; [15, 19].$$

Proof: (i) Let $a = (a_k) \in E_0$ and $x = (x_k) \in [c_0(p)]^{\alpha}$. Then for every $N > 1$,

$$|a_k| \, N^{\frac{1}{p_k}} \leq K \implies |a_k| \leq K \; N^{-\frac{1}{p_k}}$$

for all k and some $K > 0$, and

$$\sum_{k=1}^{\infty} |x_k| \, N^{-\frac{1}{p_k}} < \infty$$

for some $N > 1$. Hence

$$|a_k \; x_k| = |a_k| \, |x_k| \leq K \; |x_k| \, N^{-\frac{1}{p_k}}$$

then adding this inequality it implies that

$$\begin{aligned} \sum_{k=1}^{\infty} |a_k \; x_k| &\leq \sum_{k=1}^{\infty} K \; |x_k| \, N^{-\frac{1}{p_k}} \\ &= K \sum_{k=1}^{\infty} |x_k| \, N^{-\frac{1}{p_k}} < \infty, \end{aligned}$$

consequently $a \in [c_0(p)]^{\alpha\alpha}$, whence $E_0 \subset [c_0(p)]^{\alpha\alpha}$. Since $[c_0(p)]^{\alpha} = [c_0(p)]^{\beta}$ by Theorem 3.3 (i), it follows that $[c_0(p)]^{\alpha\alpha} \subset [c_0(p)]^{\beta\beta}$. Theorem 2 in [15], shows that

$$\lambda_1 = \bigcap_{N>1} \left\{ y = (y_k) \in \omega : \sup_k |y_k| \, N^{r_k} < \infty, \; r_k = \frac{1}{p_k} \right\}$$

replacing (y_k) by (x_k) and λ_1 by $[c_0(p)]^{\beta\beta}$, it gives us $[c_0(p)]^{\beta\beta} = E_0$, then $[c_0(p)]^{\alpha\alpha} \subset E_0$. So we get $[c_0(p)]^{\alpha\alpha} = E_0$.

(ii) It's similar to the above proof of (i), so we omit it.

Lemma 3.5: Let (p_n) be an increasing monotonic sequence of positive numbers.

(i) If $\sup_n \left| \sum_{v=1}^{n} p_v \, a_v \right| < \infty$, then $\sup_n \left| p_n \sum_{k=n+1}^{\infty} a_k \right| < \infty$,

(ii) If $\sum_{k=1}^{\infty} p_k a_k$ is convergent, then $\lim_{n \longrightarrow \infty} p_n \sum_{k=n+1}^{\infty} a_k = 0$ [16].

Theorem 3.6: For every sequence $p = (p_k)$ of strictly positive real numbers, and for every $u \in U$, then we have

(i) $[\ell_\infty(u, \triangle, p)]^\alpha = M_\alpha(p)$,

(ii) $[c_0(u, \triangle, p)]^\alpha = D_\alpha(p)$,

(iii) $[c(u, \triangle, p)]^\alpha = D\ (p)$,

where

$$\begin{aligned} M_\alpha(p) &= \bigcap_{N=2}^{\infty} \left\{ a = (a_k) \in \omega : \sum_{k=1}^{\infty} |a_k| \sum_{j=1}^{k-1} N^{\frac{1}{p_j}} w_j < \infty \right\}, \\ D_\alpha(p) &= \bigcup_{N=2}^{\infty} \left\{ a = (a_k) \in \omega : \sum_{k=1}^{\infty} |a_k| \sum_{j=1}^{k-1} N^{-\frac{1}{p_j}} w_j < \infty \right\}, \end{aligned}$$

and

$$D(p) = D_\alpha(p) \bigcap \left\{ a = (a_k) \in \omega : \sum_{k=1}^{\infty} |a_k| \sum_{j=1}^{k-1} w_j < \infty \right\} \; [14] .$$

Proof: **(i)** Let $a = (a_k)$ be an arbitrary sequence in $M_\alpha(p)$ and $x = (x_k)$ be in $\ell_\infty(u, \triangle, p)$. For any $N > \max\{1, \sup_k |u_k \triangle x_k|^{p_k}\}$, clearly

$$\sum_{k=1}^{\infty} |a_k| \sum_{j=1}^{k-1} N^{\frac{1}{p_j}} w_j < \infty \quad \text{implies} \quad \sum_{k=1}^{\infty} |a_k| < \infty.$$

On the other hand if

$$\sup_k |u_k \triangle x_k|^{p_k} < N,$$

we have

$$|\triangle x_k| < N^{\frac{1}{p_k}} w_k. \tag{3.2}$$

For each $x = (x_k)$, it gives

$$\begin{aligned} \sum_{j=1}^{k-1} \triangle x_j &= \sum_{j=1}^{k-1} (x_j - x_{j+1}) \\ &= (x_1 - x_2) + (x_2 - x_3) + ... + (x_{k-1} - x_k) \\ &= x_1 - x_k \end{aligned}$$

and hence

$$x_k = x_1 - \sum_{j=1}^{k-1} \triangle x_j \quad \text{for} \quad k = 1, 2, ... \tag{3.3}$$

By using (3.2) and (3.3) in next inequality, we have

$$\begin{aligned}
\sum_{k=1}^{\infty} |a_k x_k| &= \sum_{k=1}^{\infty} \left| a_k \left(x_1 - \sum_{j=1}^{k-1} \triangle x_j \right) \right| \\
&= \sum_{k=1}^{\infty} \left| a_k x_1 - a_k \sum_{j=1}^{k-1} \triangle x_j \right| \\
&\leq \sum_{k=1}^{\infty} |a_k x_1| + \sum_{k=1}^{\infty} \left| a_k \sum_{j=1}^{k-1} \triangle x_j \right| \\
&= \sum_{k=1}^{\infty} |a_k| \left| \sum_{j=1}^{k-1} \triangle x_j \right| + |x_1| \sum_{k=1}^{\infty} |a_k| \\
&\leq \sum_{k=1}^{\infty} |a_k| \sum_{j=1}^{k-1} N^{\frac{1}{p_j}} w_j + |x_1| \sum_{k=1}^{\infty} |a_k| < \infty. \qquad (3.4)
\end{aligned}$$

So, we have $a \in [\ell_\infty(u, \triangle, p)]^\alpha$. Thus $M_\alpha(p) \subset [\ell_\infty(u, \triangle, p)]^\alpha$. Conversely suppose that $a \notin M_\alpha(p)$. Then

$$\sum_{k=1}^{\infty} |a_k| \sum_{j=1}^{k-1} N^{\frac{1}{p_j}} w_j = \infty$$

for some $N > 1$. If we put

$$x_k = \sum_{j=1}^{k-1} N^{\frac{1}{p_j}} w_j \quad \text{for} \quad k = 1, 2, ...,$$

then x belongs to $\ell_\infty(u, \triangle, p)$ and

$$\sum_{k=1}^{\infty} |a_k x_k| = \infty.$$

This means that $a \notin [\ell_\infty(u, \triangle, p)]^\alpha$. Hence $[\ell_\infty(u, \triangle, p)]^\alpha \subset M_\alpha(p)$ and therefore $[\ell_\infty(u, \triangle, p)]^\alpha = M_\alpha(p)$.

(ii) Let $a = (a_k)$ be an arbitrary sequence in $D_\alpha(p)$ and $x = (x_k)$ be in $c_0(u, \triangle, p)$. Then there exists an integer k_0 such that

$$\sup_{k > k_0} |u_k \triangle x_k|^{p_k} \leq \frac{1}{N},$$

where N is the number in $D_\alpha(p)$. We here put M , m and L as below

$$M = \max_{1 < k < k_0} |u_k \triangle x_k|^{p_k}, \quad m = \min_{1 < k < k_0} p_k, \; L = (M+1)N$$

and define

$$y = (y_k) = \frac{x_k}{L^{\frac{1}{m}}} \quad \text{for} \quad k = 1, 2, \ldots.$$

Then it's clear to see that

$$\sup_k |u_k \bigtriangleup y_k|^{p_k} \leq \frac{1}{N},$$

and from (3.4) with N replaced by $\frac{1}{N}$, we have

$$\sum_{k=1}^{\infty} |a_k x_k| = L^{\frac{1}{m}} \sum_{k=1}^{\infty} |a_k y_k| < \infty.$$

Conversely let a does not belong to $D_\alpha(p)$. Then, we can choose an increasing sequence $(k(s))$ of integers such that $k(1) = 1$, (i.e., $k(1) < k(2) < \ldots < k(s+1) < \ldots$) and

$$M(p, s) = \sum_{k=k(s)}^{k(s+1)-1} |a_k| \sum_{j=1}^{k-1} w_j \ (s+1)^{-\frac{1}{p_j}} > 1 \quad \text{for} \quad s = 1, 2, \ldots.$$

If we take

$$x_k = \sum_{\ell=1}^{s-1} \sum_{j=k(\ell)}^{k(\ell+1)-1} w_j \ (\ell+1)^{-\frac{1}{p_j}} + \sum_{j=k(s)}^{k-1} w_j \ (\ell+1)^{-\frac{1}{p_j}},$$
$$(k(s) \leq k \leq k(s+1)-1 \quad \text{for} \quad s = 1,2,\ldots)$$

then

$$|u_k \bigtriangleup x_k|^{p_k} = \frac{1}{s+1} \quad (k(s) \leq k \leq k(s+1) - 1 \quad \text{for} \quad s = 1, 2, \ldots)$$

and hence $x \in c_0(u, \bigtriangleup, p)$. Then we may see that

$$\sum_{k=1}^{\infty} |a_k x_k| \geq \sum_{s=1}^{\infty} 1 = \infty.$$

This means that a does not belong to $[c_0(u, \bigtriangleup, p)]^\alpha$. Therefore we obtain the proof.

(iii) Let $a = (a_k)$ be an arbitrary sequence in $D(p)$ and $x = (x_k)$ be in $c(u, \bigtriangleup, p)$. Then $|u_k \bigtriangleup x_k - \ell|^{p_k} \longrightarrow 0$ $(k = 1, 2\ldots)$, for some $\ell \in \mathbb{C}$. We define $y = (y_k)$ as a sequence by

$$y_k = x_k + \ell \sum_{j=1}^{k-1} u_j^{-1}, \quad \text{for} \quad k = 1, 2, \ldots.$$

Since

$$
\begin{aligned}
|u_k \triangle y_k|^{p_k} &= |u_k (y_k - y_{k+1})|^{p_k} \\
&= \left| u_k \left[(x_k + \ell \sum_{j=1}^{k-1} u_j^{-1}) - (x_{k+1} + \ell \sum_{j=1}^{k} u_j^{-1}) \right] \right|^{p_k} \\
&= \left| u_k x_k + u_k.\ell \sum_{j=1}^{k-1} u_j^{-1} - u_k x_{k+1} - u_k.\ell \sum_{j=1}^{k} u_j^{-1} \right|^{p_k} \\
&= \left| u_k (x_k - x_{k+1}) + \ell \ u_k \left(\sum_{j=1}^{k-1} u_j^{-1} - \sum_{j=1}^{k} u_j^{-1} \right) \right|^{p_k} \\
&= \left| u_k \triangle x_k - \ell \ u_k.u_k^{-1} \right|^{p_k} \\
&= |u_k \triangle x_k - \ell \ |^{p_k} \longrightarrow 0 \quad \text{as} \quad k \longrightarrow \infty,
\end{aligned}
$$

then $y = (y_k)$ belongs to $c_0(u, \triangle, p)$. Since a belongs to $D(p)$, then by (ii), we have

$$
\begin{aligned}
\sum_{k=1}^{\infty} |a_k x_k| &= \sum_{k=1}^{\infty} \left| a_k \left(y_k - \ell \sum_{j=1}^{k-1} u_j^{-1} \right) \right| \\
&= \sum_{k=1}^{\infty} \left| a_k y_k - \ell \ a_k \sum_{j=1}^{k-1} u_j^{-1} \right| \\
&\leq \sum_{k=1}^{\infty} |a_k y_k| + |\ell| \sum_{k=1}^{\infty} |a_k| \left| \sum_{j=1}^{k-1} u_j^{-1} \right| \\
&\leq \sum_{k=1}^{\infty} |a_k| \left| \sum_{j=1}^{k-1} \triangle y_j \right| + |y_1| \sum_{k=1}^{\infty} |a_k| + |\ell| \sum_{k=1}^{\infty} |a_k| \sum_{j=1}^{k-1} w_j < \infty,
\end{aligned}
$$

and therefore $a \in [c(u, \triangle, p)]^{\alpha}$. Now suppose a belongs to $[c(u, \triangle, p)]^{\alpha}$. Since $[c(u, \triangle, p)]^{\alpha} \subset [c_0(u, \triangle, p)]^{\alpha}$ and $[c_0(u, \triangle, p)]^{\alpha} = D_{\alpha}(p)$ by (ii), then $a \in D_{\alpha}(p)$. If we here put

$$x_k = \sum_{j=1}^{k-1} w_j \quad \text{for} \quad k = 1, 2, ...,$$

then $x \in c(u, \triangle, p)$ and thus

$$\sum_{k=1}^{\infty} |a_k| \sum_{j=1}^{k-1} w_j < \infty.$$

Therefore $a \in D(p)$.

Theorem 3.7: For every sequence $p = (p_k)$ of strictly positive real numbers, and for every $u \in U$, then we have

(i) $[\ell_{\infty}(u, \triangle, p)]^{\alpha\alpha} = M_{\alpha\alpha}(p)$,

(ii) $[c_0(u, \triangle, p)]^{\alpha\alpha} = D_{\alpha\alpha}(p)$, where

$$M_{\alpha\alpha}(p) = \bigcup_{N=2}^{\infty}\left\{a=(a_k)\in\omega : \sup_{k\geq 2}|a_k|\left(\sum_{j=1}^{k-1}N^{\frac{1}{p_j}}w_j\right)^{-1}<\infty\right\},$$

and

$$D_{\alpha\alpha}(p) = \bigcap_{N=2}^{\infty}\left\{a=(a_k)\in\omega : \sup_{k\geq 2}|a_k|\left(\sum_{j=1}^{k-1}N^{\frac{-1}{p_j}}w_j\right)^{-1}<\infty\right\}\ [14].$$

Proof: **(i)** Let $a=(a_k)$ be an arbitrary sequence in $M_{\alpha\alpha}(p)$ and $x=(x_k)$ be in $[\ell_\infty(u,\triangle,p)]^{\alpha}=M_\alpha(p)$. Then we have

$$\begin{aligned}\sum_{k=2}^{\infty}|a_kx_k| &= \sum_{k=2}^{\infty}|a_k|\left(\sum_{j=1}^{k-1}N^{\frac{1}{p_j}}w_j\right)^{-1}|x_k|\sum_{j=1}^{k-1}N^{\frac{1}{p_j}}w_j\\ &\leq \left[\sup_{k\geq 2}|a_k|\left(\sum_{j=1}^{k-1}N^{\frac{1}{p_j}}w_j\right)^{-1}\right]\sum_{k=2}^{\infty}|x_k|\sum_{j=1}^{k-1}N^{\frac{1}{p_j}}w_j<\infty\end{aligned}$$

for some $N>1$, by Theorem 3.6 (i). Hence a belongs to $[\ell_\infty(u,\triangle,p)]^{\alpha\alpha}$.

Conversely suppose that a does not belong to $M_{\alpha\alpha}(p)$. Then for every integer $N>1$, we have

$$\sup_{k\geq 2}|a_k|\left(\sum_{j=1}^{k-1}N^{\frac{1}{p_j}}w_j\right)^{-1}=\infty.$$

Thus we can choose a strictly increasing sequence $(k(i))$ of integer numbers such that

$$\left|a_{k(i)}\right|\left(\sum_{j=1}^{k(i)-1}N^{\frac{1}{p_j}}w_j\right)^{-1}>i^{2}\quad \text{for } i=1,2,....$$

It implies

$$\left|a_{k(i)}\right|^{-1}\sum_{j=1}^{k(i)-1}N^{\frac{1}{p_j}}w_j<i^{-2}\quad \text{for } i=1,2,....$$

We put

$$x_k=\begin{cases}\frac{1}{\left|a_{k(i)}\right|} &, \ k=k(i)\\ 0 &, \ k\neq k(i).\end{cases}$$

Then

$$\sum_{k=1}^{\infty}|x_k|\sum_{j=1}^{k-1}N^{\frac{1}{p_j}}w_j=\sum_{i=1}^{\infty}\frac{1}{\left|a_{k(i)}\right|}\sum_{j=1}^{k(i)-1}N^{\frac{1}{p_j}}w_j\leq\sum_{i=1}^{\infty}i^{-2}<\infty$$

for every $N > 1$. This produces that $x \in [\ell_\infty(u, \triangle, p)]^\alpha$ and

$$\sum_{k=1}^{\infty} |a_k x_k| = \sum_{i\ =1}^{\infty} 1 = \infty$$

so $a \notin [\ell_\infty(u, \triangle, p)]^{\alpha\alpha}$. Hence $[\ell_\infty(u, \triangle, p)]^{\alpha\alpha} = M_{\alpha\alpha}(p)$.

(ii) Let us define

$$E_N(p) = \left\{ a = (a_k) \in \omega : \sum_{k=1}^{\infty} |a_k| \sum_{j=1}^{k-1} N^{\ -\frac{1}{p_j}} w_j \ < \infty \right\}$$

and

$$F_N(p) = \left\{ a = (a_k) \in \omega : \sup_{k\ \geq 2} |a_k| \ \left(\sum_{j=1}^{k-1} N^{\ \frac{-1}{p_j}} w_j \right)^{-1} < \infty \right\}$$

where $N = 2, 3, ...$

By Theorem 3.4 (i), $[E_N(p)]^\alpha = F_N(p)$ for each $N > 1$.

Theorem 3.8: For every sequence $p = (p_k)$ of strictly positive real numbers, then we have

$$[\ell_\infty(u, \triangle, p)]^\beta = D_\infty(p),$$

where

$$D_\infty(p) = \bigcap_{N=2}^{\infty} \left\{ a \in \omega : \sum_{k=1}^{\infty} a_k \sum_{j=1}^{k-1} N^{\ \frac{1}{p_j}} w_j \quad \text{converges and } \sum_{k=1}^{\infty} N^{\ \frac{1}{p_k}} w_k \, |R_k| < \infty \right\}$$

and

$$R_k = \sum_{v=k+1}^{\infty} a_v \quad \text{for} \quad k = 1, 2, ... \ [14] \, .$$

Proof: Let $a = (a_k)$ be an arbitrary sequence in $D_\infty(p)$ and $x = (x_k)$ be in $\ell_\infty(u, \triangle, p)$, we write

$$\sum_{k=1}^{n} a_k x_k = -\sum_{k=1}^{n-1} \triangle x_k R_k + R_n \sum_{k=1}^{n-1} \triangle x_k + x_1 \sum_{k=1}^{n} a_k \quad \text{for} \ \ n = 1, 2, ... \tag{3.5}$$

Clearly $a \in D_\infty(p)$ means that the series

$$\sum_{k=1}^{\infty} a_k$$

is convergent. Since x belongs to $\ell_\infty(u, \triangle, p)$ we may choose $N > \max\{1, \sup_k |u_k \triangle x_k|^{p_k}\}$, so that

$$\sum_{k=1}^{\infty} |\triangle x_k| \, |R_k| \leq \sum_{k=1}^{\infty} N^{\ \frac{1}{p_k}} w_k \, |R_k| < \infty.$$

Thus

$$\sum_{k=1}^{\infty} \triangle x_k R_k$$

is absolutely convergent. From Lemma 3.5 (ii), the convergence of the series

$$\sum_{k=1}^{\infty} a_k \sum_{j=1}^{k-1} N^{\frac{1}{p_j}} w_j$$

implies that

$$R_k \sum_{j=1}^{k-1} N^{\frac{1}{p_j}} w_j$$

converges to zero as $(k \longrightarrow \infty)$, and hence (3.5) produces that

$$\sum_{k=1}^{\infty} a_k x_k$$

is convergent. Thus $a \in [\ell_\infty(u, \triangle, p)]^\beta$.

Conversely suppose that $a \in [\ell_\infty(u, \triangle, p)]^\beta$. Since $e = \{1, 1, 1, ...\} \in \ell_\infty(u, \triangle, p)$, then we have the series

$$\sum_{k=1}^{\infty} a_k$$

is convergent. Define

$$x = (x_k) = \sum_{j=1}^{k-1} N^{\frac{1}{p_j}} w_j,$$

then $x \in \ell_\infty(u, \triangle, p)$, and

$$\sum_{k=1}^{\infty} a_k \sum_{j=1}^{k-1} N^{\frac{1}{p_j}} w_j$$

is convergent. By Lemma 3.5 (ii), we have the series

$$R_k \sum_{j=1}^{k-1} N^{\frac{1}{p_j}} w_j$$

converges to zero as $(k \longrightarrow \infty)$. By (3.5) we have that

$$\sum_{k=1}^{\infty} \triangle x_k R_k$$

converges for all $a \in \ell_\infty(u, \triangle, p)$. Since $x \in \ell_\infty(u, \triangle, p)$ iff $(y_k) = \{u_k \triangle x_k\} \in \ell_\infty(p)$, then $\left(u_k^{-1} R_k\right) \in [\ell_\infty(p)]^\beta$. By Theorem 3.3 (ii), we have

$$\sum_{k=1}^{\infty} N^{\frac{1}{p_k}} w_k |R_k| < \infty$$

for every $N > 1$. Hence $a \in D_\infty(p)$.

4. CONCLUSION

In this thesis, the main goal is study on Köthe-Toeplitz duals ($\alpha-$ and $\beta-$ duals) of the sequence sets $\ell_\infty(u,\triangle,p)$, $c_0(u,\triangle,p)$ and $c(u,\triangle,p)$ and focus on some results which are concluded from $\alpha-$ and $\beta-$ duals of these sequence sets. Suppose that λ represents each of the sequence sets which are in the table below, then we see the concluded results that,

λ	λ^α	$\lambda^{\alpha\alpha}$	λ^β
$\ell_\infty(u,\triangle,p)$	$M_\alpha(p)$	$M_{\alpha\alpha}(p)$	$D_\infty(p)$
$c_0(u,\triangle,p)$	$D_\alpha(p)$	$D_{\alpha\alpha}(p)$	....
$c(u,\triangle,p)$	$D(p)$	....	....

$$\begin{aligned}
M_\alpha(p) &= \bigcap_{N=2}^{\infty}\left\{a=(a_k)\in\omega:\sum_{k=1}^{\infty}|a_k|\sum_{j=1}^{k-1}N^{\frac{1}{p_j}}w_j<\infty\right\},\\
D_\alpha(p) &= \bigcup_{N=2}^{\infty}\left\{a=(a_k)\in\omega:\sum_{k=1}^{\infty}|a_k|\sum_{j=1}^{k-1}w_j\,N^{-\frac{1}{p_j}}<\infty\right\},\\
D(p) &= D_\alpha(p)\bigcap\left\{a=(a_k)\in\omega:\sum_{k=1}^{\infty}|a_k|\sum_{j=1}^{k-1}w_j<\infty\right\},\\
M_{\alpha\alpha}(p) &= \bigcup_{N=2}^{\infty}\left\{a=(a_k)\in\omega:\sup_{k\geq 2}|a_k|\left(\sum_{j=1}^{k-1}N^{\frac{1}{p_j}}w_j\right)^{-1}<\infty\right\},\\
D_{\alpha\alpha}(p) &= \bigcap_{N=2}^{\infty}\left\{a=(a_k)\in\omega:\sup_{k\geq 2}|a_k|\left(\sum_{j=1}^{k-1}N^{-\frac{1}{p_j}}w_j\right)^{-1}<\infty\right\},\\
D_\infty(p) &= \bigcap_{N=2}^{\infty}\left\{a\in\omega:\sum_{k=1}^{\infty}|a_k|\sum_{j=1}^{k-1}N^{\frac{1}{p_j}}w_j \text{ converges and } \sum_{k=1}^{\infty}N^{\frac{1}{p_k}}w_k\,|R_k|<\infty\right\}.
\end{aligned}$$

REFERENCES

[1] **Maddox, I. J.**, 1970, Elements of Functional Analysis, Cambridge University Press.

[2] **Chandrasekhara, K. Rao.**, 2006, Functional Analysis and Applications, Alpha Science International Ltd., Oxford, U.K..

[3] **Başar, F**., 2012, Summability Theory and Its Applications, Bentham Science Publishers, e-books, Monographs, Istanbul.

[4] **Kreyszig, E**., 1978, Introductory Functional Analysis with Applications, John Wiley, sons. Inc., Canada.

[5] **Trench, William F.**, 2003, Introduction to Real Analysis, Library of Congress Cataloging-in-Publication Data, San Antonio, TX, USA.

[6] **Debnath, L., Mikusinski, P.**, 2005, Hilbert Spaces with Applications, Elservier Inc., San Diego, California, USA, London, UK.

[7] **Choudhary, B., Nanda, S.**, 1987, Functional Analysis with Applications, John Wiely, Sons, New Delhi, India.

[8] **Maddox, I. J.**, 1968, On Kuttner's Theorem, J. London Math. Soc., 285 290.

[9] **Simons, S.**, 1965, The Sequence Spaces ℓ(pv) and m(pv), Proc. London Math. Soc. (3), 15, 422-436.

[10] **Alsaedi, R. S.**, **Bataineh, A. H**., 2007, On a Generalized Difference Sequence Spaces Defined by a Sequence of Orlicz Function, International Mathematical Forum, 2, no. 4, 167-177.

[11] **Maddox, I. J.**, 1969, Some Properties of Paranormed Sequence Spaces, London J., Math. Soc. (2)1, 316-322.

[12] **Nakano, H.**, 1951, Modulared Sequence Spaces, Proc. Japan Acad. 27(2), 508 512.

[13] **Malkowsky, E., Rakoĉević, V.,** 2000, An Introduction into the Theory of Sequence Spaces and Measures of Noncompactness, Zbornik Radove, Matematiči Institut SANU. Belgrade, 9 (17), 143-234.

[14] **Asma, Ç., Çolak, R.**, 2000, On the Köthe-Toeplitz Duals of Some Generalized Sets of Difference Sequences, Demonstratio Mathematica, Vol. 33, No. 4, 797-803.

[15] **Lascarides, C. G.**, 1971, A Study of Certain Sequence Spaces of Maddox and a Generalization of a Theorem of Iyer, Pacific J. Math. Vol. 38, 487-500.

[16] **Kizmaz, H.,** 1981, On Certain Sequence Spaces, Canadian Math. Bull. 24, 169-176.

[17] **Lascarides, C. G., Maddox, I. J.,** 1970, Matrix Transformations Between Some Classes of Sequences, Proc. Cambridge Phil. Soc. 68, 99-104.

[18] **Aiyub, M.**, 2008, Some Paranormed Sequence Spaces and Matrix Transformations, Int. J. Contempt, Math. Sciences, Vol. 3, no. 10, 463-469.

[19] **Çolak, R., Srivastava, P. D., Nanda, S.,** 1993, On Certain Sequence Spaces and Their Köthe-Toeplitz Duals, Rendiconti di Matematica, Serie VII, Volueme 13, Roma, 27-29.

[20] **Maddox, I. J.,** 1969, Continuous and Köthe-Toeplitz Duals of Certain Sequence Spaces, proc. Cambridge Philos. Soc., 65, 431-435.

[21] **Ahmad, Z. U., Mursaleen, M.**, 1987, Köthe-Toeplitz Duals of Some New Sequence Spaces and Their Matrix Maps, Publ Inst Math (Beograd), 42: 57-61.

[22] **Altay, B., Başar, F.,** 2007, Certain Topological Properties and Duals of The Matrix Domain of a Triangle Matrix in a Sequence Space, J. Math Anal Appl, 336(1): 632-645.

[23] **Malkowsky, E.,** 1996 A Note On The Köthe-Toeplitz Duals of Generalized Sets of Bounded and Convergent Difference Sequences, J. Anal. 4, 81-91.

[24] **Et, M.**, 1993, On Some Difference Sequence Spaces, Doga-Tr. J. Math. 17, 18-24.

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