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An Analytical Method for Computing the Coefficients of the Trivariate Polynomial Approximation

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Abstract

This paper deals with computing the coefficients of the trivariate polynomial approximation (TPA) of large distinct points given on $\Re^3$. The TPA is formulated as a matrix equation using Kronecker and Khatri-Rao products of the matrices. The coefficients of the TPA are computed using the generalized inverse of the matrix. It is seen that the trivariate polynomial approximation can be investigated as the matrix equation and the coefficients of the TPA can be computed directly from the solution of the matrix equation.

Keywords: Polynomial approximation, Trivariate polynomial, Generalized inverse of a matrix, Matrix equation

Üç Değişkenli Polinom Yaklaşımının Katsayılarının Hesabı İçin Bir Analitik Yöntem

Özet

Bu makale, $\Re^3$de verilen çok sayıda farklı noktanın üç değişkenli polinom yaklaşımındaki katsayıların hesaplaması ile ilgilidir. Üç değişkenli polinom yaklaşımı, matrislerin Kronecker ve Khatri - Rao çarpımlarını kullanarak bir matris denklemi olarak formüle edilmiştir. Bu polinomun katsayıları bir matrisin genelleştirilmiş tersi kullanılarak hesaplanmıştır. Üç değişkenli polinom yaklaşımın bir matris denklemi olarak incelenebileceği ve bu polinomun katsayılarının doğrudan bu matris denkleminin çözümünden hesaplanabileceği görülmüştür.

Anahtar kelimeler: Polinom yaklaşımı, Üç değişkenli polinom, Matrisin genelleştirlmiş tersi, Matris denklemi.

1.Introduction

One of the most interesting mathematical studies is that finding the best polynomial approximation of the given data or the function [1-11]. Multivariable approximation and interpolation has been an active research area in applied mathematics for many years, and has had impact on various engineering and scientific applications in mathematical modeling, in computations with large scale data, in signal analysis and image processing [4, 6, 8, 10-12].

The polynomial approximation is investigated using the different forms and algorithms. Also, it is solvable either numerically or using a computer algebra package [1, 3, 7]. The polynomial approximations in two or several variables is the most commonly investigated problem by researchers [2, 4, 5, 10]. There are many research for the developments of least squares polynomial approximations to sets of data and the polynomial approximations in several variables have been investigated by the different methods [1, 8-11, 13, 14].

In this study, we formulate the trivariate polynomial approximation (TPA) as a matrix equation using Kronecker and Khatri-Rao products of the matrices and compute the coefficients of the trivariate polynomial approximation using the generalized inverse matrices. It is seen that the trivariate polynomial approximation can be investigated as the matrix equation and the coefficients of the TPA can be computed directly from the solution of the matrix equation.

2.Basic definitions and theorems

In this section, we give some basic definitions and theorems associated with the trivariate polynomial, Kronecker product, Khatri-Rao product and generalized inverse of a matrix. Further details and proofs can be found elsewhere [1-3, 5, 6, 14-19].

Definition 1. Given a hyper surface $u = h(x, y, z)$ on $\Re^4$ to approximate over a region that is gridded by (x_l, y_l, z_{k_l}), $0 \le l \le s$, $0 \le k_l \le r_l$ on $\Re^3$. Assumed that $u_{llk_l} = h(x_l, y_l, z_{k_l})$ data are given for the function of three variables at the $(s+1)\sum_{l=0}^{s}(r_l+1)$ distinct points in the

region on $\Re^3$, there is a hyper surface on $\Re^4$ of the form

$$p(x, y, z) = \sum_{i=0}^{m}\sum_{j=0}^{n}\sum_{k=0}^{r} a_{ijk} x^i y^j z^k \tag{1}$$

of degree, not exceding $m+n+r$, namely $x^m y^n z^r$ that passes through each point in the solid region, where $(m+1)(n+1) \le (s+1)$ and $r \le \min\{r_i\}$ for $i = 0,1,2,\ldots,r$. We say that Eq.(1) is a trivariate polynomial approximation satisfiying $p(x_l, y_l, z_{k_l}) \cong h(x_l, y_l, z_{k_l})$ for all $0 \le l \le s$ and $0 \le k_l \le r_l$. It is clear that $h(x, y, z)$ at any point $(\hat{x}, \hat{y}, \hat{z})$, which is not in solid region on $\Re^3$, can be estimated by $h(\hat{x}, \hat{y}, \hat{z}) \cong p(\hat{x}, \hat{y}, \hat{z})$ [5-7].

Now we define the trivariate polynomial approximation for general distinct points. Given $u = f(x, y, z)$ to approximate over a solid rectangular region that is gridded by (x_i, y_j, z_k) on $\Re^3$, $0 \le i \le m$, $0 \le j \le n$ and $0 \le k \le r$.

Assumed that $f_{ijk} = f(x_i, y_j, z_k)$ data are given for the function of three variables at the $(m+1)(n+1)(r+1)$ distinct points in the solid rectangular region, there is a hyper surface on $\Re^4$ of the form

$$p(x, y, z) = \sum_{i=0}^{p}\sum_{j=0}^{q}\sum_{k=0}^{t} b_{ijk} x^i y^j z^k \tag{2}$$

degree at $p+q+t$, namely $x^p y^q z^t$ that deviates as little as possible from f in the solid rectangular region, where $p \le m$, $q \le n$ and $t \le r$. Also, we say that Eq.(2) is a trivariate polynomial approximation which satisfies $p(x_i, y_j, z_k) \cong f(x_i, y_j, z_k)$ at minimum error for all $0 \le i \le p$, $0 \le j \le q$ and $0 \le k \le t$ [5-7].

Definition 2. Let $A = [a_{ij}]$ be an $m \times n$ matrix and $B = [b_{ij}]$ be a $p \times q$ matrix, then the Kronecker product of A and B, $A \otimes B$, which is an $mp \times nq$ matrix is defined by

$$A \otimes B = [Ab_{ij}]. \tag{3}$$

Let $A = [A_{ij}]$ be partinoned with A_{ij} of order $m_i \times n_j$ as the $(i, j)th$ block matrix and let $B = [B_{kl}]$ partinoned with B_{kl} of order $p_k \times q_l$ as the $(k,l)th$ block matrix where, $\sum m_i = m$, $\sum n_j = n$, $\sum p_k = p$ and $\sum q_l = q$. The Khatri–Rao product of the matrices A and B is defined as

$$A * B = [A_{ij} \otimes B_{ij}]_{ij}, \tag{4}$$

where $A_{ij} \otimes B_{ij}$ is order of $m_i p_i \times n_j q_j$ and $A * B$ of order $\left(\sum m_i p_i\right) \times \left(\sum n_j q_j\right)$ [15, 19, 20].

Definition 3. Let A be an arbitrary $m \times n$ complex matrix. The generalized inverse A^+ of A uniquely determined as the $n \times m$ matrix, which is simultaneously satisfied the following system of four matrix equations:

$$A^+AA^+ = A^+,\ AA^+A = A,$$
$$(AA^+)^* = AA^+,\ (A^+A)^* = A^+A. \tag{5}$$

If A is a real matrix, $A^* = A^T$ where A^* denotes the conjugate transpose of A [14, 16-18].

Theorem 1. A necessary and sufficient condition for the equation $XBY = C$ to have a solution is that

$$XX^+CY^+Y = C$$

in which case the general solution is

$$B = X^+CY^+ + W - X^+XWYY^+,$$

where W is an arbitrary matrix and X^+ is the generalized inverse of X [14,18].

3.Trivariate polynomial approximation

In the this section, we first formulate a matrix equation to calculate the coefficients of the trivariate polynomial approximation defined in Eqs.(1) and (2) using Kronecker and Khatri-Rao products, and then investigate the solution of this matrix equation using the generalized inverse of a matrix.

Let

$$\begin{aligned} \boldsymbol{x} &= [1 \quad x \quad x^2 \quad \ldots \quad x^m], \\ \boldsymbol{y} &= [1 \quad y \quad y^2 \quad \ldots \quad y^n], \\ \boldsymbol{z} &= [1 \quad z \quad z^2 \quad \ldots \quad z^r] \end{aligned} \tag{6}$$

and

$$A^T = [A_0 \quad A_1 \quad \ldots \quad A_m], \tag{7}$$

where

$$A_0 = \begin{bmatrix} a_{000} & a_{010} & \cdots & a_{0n0} \\ a_{001} & a_{011} & \cdots & a_{0n1} \\ \vdots & \vdots & \vdots & \vdots \\ a_{00r} & a_{01r} & \cdots & a_{0nr} \end{bmatrix}$$

$$A_1 = \begin{bmatrix} a_{100} & a_{110} & \cdots & a_{1n0} \\ a_{101} & a_{111} & \cdots & a_{1n1} \\ \vdots & \vdots & \vdots & \vdots \\ a_{10r} & a_{11r} & \cdots & a_{1nr} \end{bmatrix}, \ldots,$$

$$A_m = \begin{bmatrix} a_{m00} & a_{m10} & \cdots & a_{mn0} \\ a_{m01} & a_{m11} & \cdots & a_{mn1} \\ \vdots & \vdots & \vdots & \vdots \\ a_{m0r} & a_{m1r} & \cdots & a_{mnr} \end{bmatrix},$$

$A_0, A_1, \ldots, A_m$ are $(r+1)\times(n+1)$ matrices and A is the $(m+1)(n+1)\times(r+1)$ matrix. Using (6) and (7), we can state the trivariate polynomial defined in Eq.(1) as a matrix equation as follows:

$$p(x, y, z) = (\boldsymbol{y} \otimes \boldsymbol{x}) A \boldsymbol{z}^T, \tag{8}$$

where $\otimes$ is the Kronecker product. Eq.(8) is the matrix equation form of Eq. (1).

The polynomial approximation in three variables defined in Eq.(8) can be found such that the equation must satisfy $p(x_l, y_l, z_{k_l}) \cong h(x_l, y_l, z_{k_l})$ for all $0 \le l \le s$, and $0 \le k_l \le r_l$ at minimum error. For this purpose, we can compute the matrix A from a matrix equation.

Let

$$M = \begin{bmatrix} \boldsymbol{y}_0 \otimes \boldsymbol{x}_0 & 0 & \ldots & 0 \\ 0 & \boldsymbol{y}_1 \otimes \boldsymbol{x}_1 & \ldots & 0 \\ \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & \cdots & \boldsymbol{y}_s \otimes \boldsymbol{x}_s \end{bmatrix},$$

$$Z = \begin{bmatrix} Z_0 & 0 & \ldots & 0 \\ 0 & Z_1 & \ldots & 0 \\ \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & \cdots & Z_s \end{bmatrix}, \tag{9}$$

$$F = \begin{bmatrix} \boldsymbol{u}_0 & 0 & \ldots & 0 \\ 0 & \boldsymbol{u}_1 & \ldots & 0 \\ \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & \cdots & \boldsymbol{u}_s \end{bmatrix},$$

where

$$\boldsymbol{y}_l \otimes \boldsymbol{x}_l = [1 \quad y_l \quad {y_l}^2 \quad \ldots \quad {y_l}^n] \otimes [1 \quad x_l \quad {x_l}^2 \quad \ldots \quad {x_l}^m]$$

and

$$\boldsymbol{u}_l = [u_{ll0_l} \quad u_{ll1_l} \quad \ldots \quad u_{llr_l}]$$

for $l = 0, 1, \ldots, s$ and

$$Z_l = \begin{bmatrix} 1 & 1 & \dots & 1 \\ z_{0_l} & z_{1_l} & \dots & z_{r_l} \\ \vdots & \vdots & \dots & \vdots \\ z_{0_l}^r & z_{1_l}^r & \dots & z_{r_l}^r \end{bmatrix},$$

where M is the $(s+1)\times(s+1)(m+1)(n+1)$ matrix of rank $(s+1)$, Z is the $(r+1)(s+1)\times\sum_{l=0}^{s}(r_l+1)$ matrix of rank $(s+1)(r+1)$, F is the $(s+1)\times\sum_{i=0}^{s}(r_l+1)$ matrix and $\boldsymbol{u}_l$ is the $1\times(r_l+1)$ vector.

Using (9) and properties of Kronecker product of matrices, we can formulate the coefficients of Eq.(1) as the matrix equation for all $l = 0, 1, 2, \dots, s$ as follows:

$$M(A\otimes I_{s+1})Z = H\,, \tag{10}$$

where I_{s+1} is the $(s+1)\times(s+1)$ identity matrix.

We can compute the matrix A from Eq.(10) and we can find Eq.(1) or Eq.(6) using the matrix A which is approximated with minimum error. For this purpose, we can solve Eq.(10) using the generalized inverses of matrices. This solution is known as the best solution.

Theorem 2. Eq.(10) has a solution and its general solution is

$$A = \frac{1}{(y_i y_i^T)(x_i x_i^T)}(y_i^T\otimes x_i^T)\left(u_i Z_i^+ - (y_i\otimes x_i)W\right) + W \tag{11}$$

for $l = 0, 1, 2, \dots, s$, where W is an arbitrary matrix.

Proof. Let $A\otimes I_{s+1}$ be solution of (10). Since

$$\begin{aligned} H &= M(A\otimes I_{s+1})Z \\ &= MM^+M(A\otimes I_{s+1})ZZ^+Z \\ &= MM^+HZ^+Z, \end{aligned}$$

Eq.(10) has a solution, where $MM^+ = I$ and $ZZ^+ = I$. Using Theorem 1 and Eq.(11), we obtain

$$M^+ = Diag\left\{\frac{(y_i^T\otimes x_i^T)}{(y_i y_i^T)(x_i x_i^T)}\right\},$$

for $i = 0,1,2,\dots,s$,

$$Z^+ = Diag\left\{Z_0^+,\quad Z_1^+,\quad \dots,\quad Z_s^+\right\}$$

and

$$A\otimes I_{s+1} = M^+HZ^+ + (I - M^+M)W\,.$$

Eq.(11) are the approximate solutions of (10). In this study, we also compute the best approximate solution of (10). For this, we can rewrite Eq.(10), using properties of Kronecker product, Kahatri-Rao product and generalized inverse of the matrices, as

$$(V_{ys} * V_{xs})A = U\,, \tag{12}$$

where

$$U = \begin{bmatrix} \boldsymbol{u}_0 Z_0^+ \\ \boldsymbol{u}_1 Z_1^+ \\ \vdots \\ \boldsymbol{u}_s Z_s^+ \end{bmatrix}, V_{xs} = \begin{bmatrix} \boldsymbol{x}_0 \\ \boldsymbol{x}_1 \\ \boldsymbol{x}_2 \\ \vdots \\ \boldsymbol{x}_s \end{bmatrix},\ V_{ys} = \begin{bmatrix} \boldsymbol{y}_0 \\ \boldsymbol{y}_1 \\ \boldsymbol{y}_2 \\ \vdots \\ \boldsymbol{y}_s \end{bmatrix},$$

$*$ is Khatri–Rao product, $V_{ys} * V_{xs}$ is the $(s+1)\times(m+1)(n+1)$ matrix of rank $(m+1)(n+1)$, U is the $(s+1)\times(r+1)$ matrix, the rank of Z_l for $l = 0, 1, 2, \dots, s$ is $r+1$, $Z_l^+ = Z_l^T(Z_l Z_l^T)^{-1}$ and $Z_l Z_l^+ = I_{r+1}$.

Using Theorem 1, we solve the Eq.(12) and find the unique solution matrix A which is the best solution of (12). This leads to following result.

Theorem 3. The best solution of Eq.(12) is

$$A = [(V_{ys}^T * V_{xs}^T)(V_{ys} * V_{xs})^{-1}](V_{ys} * V_{xs})U\,. \tag{13}$$

Proof Let $V_{ys} * V_{xs} = V$. Since the rank of the matrix V is $(m+1)(n+1)$, $V^+ = (V^+V)^{-1}V^T$ and $V^+V = I_{(m+1)(n+1)}$. Then we have,

$U = VA = VV^{+}VA = VV^{+}U$.

Thus we see that Eq.(12) has a solution. In which case, we obtain the general solution as

$A = A_p + A_h$,

where $A_p = V^{+}U$ and $A_h = W - V^{+}VW = 0$ are the particular and homogenous solutions, respectively and W is an arbitrary matrix. If $A_p = A$, then Eq.(12) has a unique solution. Thus the proof is completed.

Also, we can construct the matrix equation to find the trivariate polynomial for $f_{ijk} = f(x_i, y_j, z_k)$ data which are given for the function of three variables at the $(m+1)(n+1)(r+1)$ distinct points in the solid rectangular region.

Let

$$B = [B_0 \quad B_1 \quad \dots \quad B_t], \tag{14}$$

where

$$B_0 = \begin{bmatrix} b_{000} & b_{100} & \cdots & b_{p00} \\ b_{010} & b_{110} & \cdots & b_{p10} \\ \vdots & \vdots & \vdots & \vdots \\ b_{0q0} & b_{1q0} & \cdots & b_{pq0} \end{bmatrix}$$

$$B_1 = \begin{bmatrix} b_{001} & b_{101} & \cdots & b_{p01} \\ b_{011} & b_{111} & \cdots & b_{p11} \\ \vdots & \vdots & \vdots & \vdots \\ b_{0q1} & b_{1q1} & \cdots & b_{pq1} \end{bmatrix}, \dots,$$

$$B_t = \begin{bmatrix} b_{00t} & b_{10t} & \cdots & b_{p0t} \\ b_{01t} & b_{11t} & \cdots & b_{p1t} \\ \vdots & \vdots & \vdots & \vdots \\ b_{0qt} & b_{1qt} & \cdots & b_{pqt} \end{bmatrix},$$

$$\begin{aligned} \boldsymbol{x} &= [1 \quad x \quad x^2 \quad \dots \quad x^p], \\ \boldsymbol{y} &= [1 \quad y \quad y^2 \quad \dots \quad y^q], \\ \boldsymbol{z} &= [1 \quad z \quad z^2 \quad \dots \quad z^t] \end{aligned} \tag{15}$$

and B_k is an $(q+1)\times(p+1)$ matrix.

Note that b_{ijk} are the elements of the $(q+1)\times(p+1)(t+1)$ matrix B defined in Eq.(2), where $0 \le i \le p$, $0 \le j \le q$ and $0 \le k \le t$. Using (14) and (15), we can state TPA defined in Eq.(2) as a matrix equation as follows:

$$p(x, y, z) = \boldsymbol{y}B(\boldsymbol{x}^T \otimes I_{t+1})\boldsymbol{z}^T \tag{16}$$

Also, we can express the coefficients of the trivariate plynomial which is the best approximate as a matrix equation such that Eq.(2) must satisfy at minimum error $p(x_i, y_j, z_k) \cong f(x_i, y_j, z_k)$ in the solid rectangular region.

Let

$$V_x = \begin{bmatrix} \boldsymbol{x}_0 \\ \boldsymbol{x}_1 \\ \boldsymbol{x}_2 \\ \vdots \\ \boldsymbol{x}_m \end{bmatrix}, \quad V_y = \begin{bmatrix} \boldsymbol{y}_0 \\ \boldsymbol{y}_1 \\ \boldsymbol{y}_2 \\ \vdots \\ \boldsymbol{y}_n \end{bmatrix}, \quad V_z = \begin{bmatrix} \boldsymbol{z}_0 \\ \boldsymbol{z}_1 \\ \boldsymbol{z}_2 \\ \vdots \\ \boldsymbol{z}_r \end{bmatrix} \tag{17}$$

and

$$F = [F_0 \quad F_1 \quad \dots \quad F_m], \tag{18}$$

where

$$F_0 = \begin{bmatrix} f_{000} & f_{001} & \cdots & f_{00r} \\ f_{010} & f_{011} & \cdots & f_{01r} \\ \vdots & \vdots & \vdots & \vdots \\ f_{0n0} & f_{0n1} & \cdots & f_{0nr} \end{bmatrix}$$

$$F_1 = \begin{bmatrix} f_{100} & f_{101} & \cdots & f_{10r} \\ f_{110} & f_{111} & \cdots & f_{11r} \\ \vdots & \vdots & \vdots & \vdots \\ f_{1n0} & f_{1n1} & \cdots & f_{1nr} \end{bmatrix}, \dots,$$

$$F_m = \begin{bmatrix} f_{m00} & f_{m01} & \cdots & f_{m0r} \\ f_{m10} & f_{m11} & \cdots & f_{m1r} \\ \vdots & \vdots & \vdots & \vdots \\ f_{mn0} & f_{mn1} & \cdots & f_{mnr} \end{bmatrix}$$

$\boldsymbol{x}_i = [1 \quad x_i \quad {x_i}^2 \quad \dots \quad {x_i}^p]$,

$\boldsymbol{y}_j = [1 \quad y_j \quad {y_j}^2 \quad \dots \quad {y_j}^q]$,

$$\boldsymbol{z}_k = [1 \quad z_k \quad {z_k}^2 \quad \ldots \quad {z_k}^t]$$

and F_i is an $(n+1)\times(r+1)$ matrix.

Note that $f(x_i, y_j, z_k) = f_{ijk}$ are the elements of the matrix F defined in Eq.(18) where $0 \le i \le m,\ 0 \le j \le n$ and $0 \le k \le r$.

Using (17) and (18), we can formulate the coefficients of Eq.(2) as follows:

$$V_y\, B\left({V_x}^T * (I_{t+1} \otimes \boldsymbol{1}_{m+1}^T)\right)\left({V_z}^T \otimes I_{m+1}\right) = F\,, \quad (19)$$

where $\boldsymbol{1}_{m+1}$ is the $(m+1)\times 1$ vector whose entries are 1.

We can find the matrix B from Eq.(19). This leads to following results.

Corollary 1. Let $\left({V_x}^T * (I_{t+1} \otimes \boldsymbol{1}_{m+1}^T)\right)$ be the matrix defined in Eq.(19). Then its generalized inverse is

$$\begin{aligned}&\left({V_x}^T * (I_{t+1} \otimes \boldsymbol{1}_{m+1}^T)\right)^+ = \\ &\quad \left(V_x * (I_{t+1} \otimes \boldsymbol{1}_{m+1})\right)\left(\left(V_x^T V_x\right)^{-1} \otimes I_{r+1}\right).\end{aligned} \quad (20)$$

Proof Observing the rank of the $(p+1)(t+1)\times(t+1)(m+1)$ matrix defined in Eq.(19) is $(p+1)(t+1)$, and using the generalized inverse and Khatri-Rao product of the matrices, we easily prove it.

Theorem 4. Eq.(19) has a unique solution as follows:

$$B = {V_y}^+ F\left({V_z}^+ \otimes I_{m+1}\right)^T \left(V_x * (I_{t+1} \otimes \boldsymbol{1}_{m+1})\right)[({V_x}^T V_x)^{-1} \otimes I_{t+1}] \quad (21)$$

Proof Using Theorem 1, Eq.(19) has a solution. Since ranks of the matrices V_x, V_y and V_z are $p+1$, $q+1$ and $t+1$ respectively, ${V_x}^+ V_x = I_{p+1}$, ${V_y}^+ V_y = I_{q+1}$ and ${V_z}^+ V_z = I_{t+1}$. In which case we obtain the general solution as $B = B_p + B_h$, where

$$\begin{aligned}&B_p = \\ &{V_y}^+ F\left({V_z}^+ \otimes I_{m+1}\right)^T \left(V_x * (I_{t+1} \otimes \boldsymbol{1}_{m+1})\right)[({V_x}^T V_x)^{-1} \otimes I_{t+1}],\end{aligned}$$

$$\begin{aligned}&B_h = \\ &W - {V_y}^+ V_y W\left({V_z}^T \otimes I_{m+1}\right)^+ \left({V_x}^T * (I_{t+1} \otimes \boldsymbol{1}_{m+1}^T)\right)^+ \left({V_x}^T * (I_{t+1} \otimes \boldsymbol{1}_{m+1}^T)\right)\left({V_z}^T \otimes I_{m+1}\right) \\ &= 0\end{aligned}$$

W is an arbitrary matrix, and B_p and B_h are the particular and homogenous solutions, respectively. If $B_p = B$, then (19) has a unique solution. Thus the proof is completed.

Theorem 5. Let B_k and F_i be submatrices defined in (14) and (18). Then, the solution of Eq.(19) is

$$B = V_y^+ \sum_{i=0}^{m} F_i \left(V_z^T\right)^+ \left[\left(\boldsymbol{x}_i (V_x^T V_x)^{-1}\right) \otimes I_{r+1}\right]. \quad (22)$$

Proof To prove the theorem, we can use Theorem 4 and the matrices defined in (14) and (18). Putting (14) and (18) in (21) and using Kronecker and Khatri-Rao products, then Eq.(22) is obtained.

4. Conclusions

We consider the trivariate polynomial approximation (TPA) of given distinct points on $\Re^3$ and formulate the TPA as a matrix equation using Kronecker and Khatri-Rao products of the matrices. We conclude that the coefficients of the TPA can be computed directly from the matrix equation by the use of generalized inverse matrices. It is seen that the TPA can be investigated as the matrix equation and the coefficients of it can be computed directly from this matrix equation.

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